

## STABILIZATION OF DISTURBED MOTION'S TRAJECTORIES IN SYMMETRIZATION OF PHASE LIMITATIONS.

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**Abstract.** One of the actual problems of the objectives of control is the problem of synthesizing the law of control, effective under the conditions of various indeterminacy and limitations on motion properties. At the same time the deriving of the robust laws of control is very important under the conditions of illegible data. In this work it is shown, that the solving of the mentioned problem comes to solution of some certain logarithmic correlations from the specially created connected phase plane. As far as smooth phase limitations are concerned, these correlations can be converted into algebraic equations, according to the synthesized law of control. The investigation in the cases of uneven limitations, which describe linear polyhedrons in state space, is performed in this work. It is shown, that in case of being symmetrical to set point of the coordinate system the solvability of synthesizing problem is all defined by the random apex of the polyhedron. If the required condition of the symmetry is not fulfilled, than the minimum amount of the random polyhedron apexes is concerned for analyzing the solvability of the objective. The description (symmetrization) of the phase plane is concerned with the help of the polyhedrons, symmetrical in any way. At the same time the system` indeterminacy is considered straight according to the algebraic correlations of the symmetric polyhedrons` apexes.

### 1. INTRODUCTION

The problem of controlling the complex objects under the conditions of various indeterminacy and limitations on motion properties is one of the most relevant problems. Many works, [1]-[5], are dedicated to its solution. Applying of known methods doesn't let to solve the problem effectively. It's concerned with complicated limitations set for this system.

Though, one of the main problems in synthesis of regulators is working out a direct method of the synthesis, which allows considering the system's limitations straight away in conditions of various indeterminacy and forming effective robust control laws. A detailed investigation of solution of this problem, along with the case of linear system with the limitations set as phase polyhedron is performed in this work. The suggested approach is based on the method of variation of the phase limitations, instead of solving the problem of synthesis basing on forming the symmetrical phase limitations. This allows not only to expand the set of the permissible solutions, but also to simplify the procedure getting them.

## 2. STATEMENT OF THE TASK

Lets take into consideration a system close to a general system type:

$$\dot{x} = f(x, u), x(t_0) = x_0, t \geq t_0 \quad (1)$$

Where  $x, u - n \times 1, m \times 1, r \times 1$  are vectors of condition and control ;  $f(\cdot)$  -  $n \times 1$  vector-function satisfying a condition of existence and uniqueness of solution of the Cauchy problem.

The restrictions are defined in the following way:

$$x(t) \in Q(t), \quad t \geq t_0, \text{ where } Q(t) = \left\{ x \in R^n : \psi_i(x, t) \leq 0, i \in \overline{1, \chi} \right\} \quad (2)$$

$\psi_i(x, t)$  - continuously differentiable functions.

Lets designate the border of the set  $Q(t)$  as  $\Gamma Q(t)$ , and as  $\Gamma Q_i(t)$  - *hyper surfaces* of the type

$$\Gamma Q_i = \left\{ x \in R^n : \psi_i(x, t) = 0 \right\}, i \in \overline{1, \chi} \quad (3)$$

Lets also consider the system in case of being prone to external disturbances and containing internal indeterminacies (parametric and structural). Then they could be defined as equation

$$\dot{x} = \varphi(z, x, u, v), \quad x(t_0) = x_0, t \geq t_0, \quad (4)$$

Where  $v - r \times 1$  is a vector of disturbance,  $z - p \times 1$  - vectorial parameter, which characterizes internal indeterminacies of the system.  $v$  and  $z$  satisfy the conditions  $v \in V, z \in Z$

Where  $V, Z$  - some given sets in  $R^r$  and  $R^p$ .

Let's consider  $u = \tilde{u}(x)$  to be a control law, which has set form (i.e. it has any set structure). Lets say, that  $\tilde{u}(x)$  has equivalent structure. Then the problem of synthesis may be formulated as following:

For the system (1) or (4) it is necessary to synthesize a control law for a desired structure  $\tilde{u}(\cdot)$ , providing the fulfillment of phase limitations (2) for all trajectories  $x(t)$ , for which  $x(t_0) \in Q(t_0)$  and possible influence of external and internal indeterminacies is taken into consideration.

The solution of the stated problem is based on the usage of some geometric properties of system trajectory.

## 3. CREATING OF "TRANSPARENT" AND "OPAQUE" AREAS IN THE STATE SPACE

Let's investigate the system (1). For each continuously differentiable limitation function  $\psi_i(x, t), i \in \overline{1, \chi}$  it is possible to form sets:

$\rho_i^- = \left\{ x \in R^n : (\nabla_x \psi_i, f(\cdot)) + \frac{\partial \psi_i}{\partial t} < 0 \right\}$  - "opaque" sets;  $\rho_i^+ = \left\{ x \in R^n : (\nabla_x \psi_i, f(\cdot)) + \frac{\partial \psi_i}{\partial t} > 0 \right\}$  -  
"transparent" sets;  $\Gamma \rho_i = \left\{ x \in R^n : (\nabla_x \psi_i, f(\cdot)) + \frac{\partial \psi_i}{\partial t} = 0 \right\}$  - border of sets  $\rho_i^-$  u  $\rho_i^+$ ; where  
 $\nabla_x \psi_i = \left[ \frac{\partial \psi_i}{\partial x_1}, \frac{\partial \psi_i}{\partial x_2}, \dots, \frac{\partial \psi_i}{\partial x_n} \right]^T$  - function  $\psi_i(x, t)$  gradient,  $(d, g)$  - scalar product in  
Euclidean arithmetic space  $R^n$

Furthermore, let's form an additional set:

$$Q_i^\delta(t) = \left\{ x \in R^n : \psi_i(x, t) \leq \delta \right\}, \delta \geq \min_x \psi_i(x, t), i \in \overline{1, \chi} \text{ with a border}$$

$$\Gamma Q_i^\delta(t) = \left\{ x \in R^n : \psi_i(x, t) = \delta \right\}, i \in \overline{1, \chi}$$

- which are the sets and surfaces of the functions' levels  $\psi_i(\cdot)$ . Then

$$Q(t) = \bigcap_{i=1}^{\chi} Q_i(t), Q_i^0(t) = Q_i(t), \Gamma Q_i^0(t), \delta = 0 \quad (5)$$

Lets say, that a set  $N^+$  is "transparent" for a set  $Q_i(t)$ , if along an arbitrary trajectory  $x(t)$ , passing through  $N^+$  at any t (i.e. at  $x(t) \in N^+$ ), function  $\psi_i(\cdot)$  is increasing (i.e. through the element of set  $N^+$  trajectory  $x(t)$  is striving for exiting from  $Q_i(t)$  or for moving away from  $Q_i(t)$ ).

So, the set  $N^-$  - "transparent" for a set  $Q_i(t)$ , if function  $\psi_i(\cdot)$  is decreasing along an arbitrary trajectory  $x(t)$  at  $x(t) \in N^-$  (i.e. through the element of set  $N^-$  trajectory  $x(t)$  cannot exit from  $Q_i(t)$  or striving for getting into it). It is clear, that for each set  $Q_i(t)$

$$N^+ = \rho_i^+, \quad N^- = \rho_i^-, i \in \overline{1, \chi} \text{ is true, as along arbitrary } x(t)$$

$$\frac{d}{dt} \psi_i(x(t), t) = \dot{\psi}_i = (\nabla_x \psi_i, f(\cdot)) + \frac{\partial \psi_i}{\partial t}$$

Arbitrary trajectory  $x(t)$  of the system (1) at  $x(t_0) = x_0 \in Q(t_0)$  will not exceed the bounds of  $Q(t)$  at  $t > t_0$ , if

$$\Gamma Q(t) \cap \Gamma Q_i(t) \subset \rho_i^-, \quad i \in \overline{1, \chi}, t \geq t_0 \quad (6)$$

#### 4. FORMING OF THE CORRELATIONS FOR SYNTHESIZING THE CONTROL LAW

Let's use conditions (6) for solving the stated problem. It is obvious, that if for some  $i \in \overline{1, \chi}$   $\Gamma Q \cap \Gamma Q_i \subset \rho_i^-$  then

$$\Gamma \rho_i \cap [\Gamma Q \cap \Gamma Q_i] = \emptyset \quad (\emptyset \text{-designation of an empty set}) \quad (7)$$

$$\Gamma \rho_i \cap [\Gamma Q \cap \Gamma Q_i] = \{S^i\} \quad (8)$$

Where  $S^i$  – is a point of contact of the surfaces  $\Gamma \rho_i$  and  $\Gamma Q \cap \Gamma Q_i$ .

From (8) follows, that the gradients of contacting(tangent) surfaces in  $S^i$  have opposite directions.

Let  $\nabla_x \psi_i, \nabla_x \dot{\psi}_i$  be the function of  $\psi_i(\cdot)$  and  $\dot{\psi}_i(\cdot)$  gradients. Obviously, (8) implements if and only if, the next equation is right:

$$\nabla_x \psi_i + \lambda_i \nabla_x \dot{\psi}_i = 0, \lambda_i > 0; x \in \Gamma Q(t) \cap \Gamma Q_i(t) \quad (9)$$

In future we will consider, that, if  $Q(t)$  is given as some phase polyhedron (set with uneven bound), then the general conditions for providing phase limitations are the correlations (7).

Obviously, the conditions (7) can be represented by

$$\psi_i(x, t) = 0; \dot{\psi}_i(x, u, t) = 0; \exists v \in \overline{1, \chi} \setminus i, \text{ that } \psi_v(x, t) > 0 \quad (10)$$

The result leads to the correctness of the following statement.

In some cases correlations (10) may be a direct solution in analytical form regarding the parameters of control law of the desired structure. However, the analytical solution is not necessary in general case. It is enough to shown, when at given  $\tilde{u}(x)$  correlations (10) are solvable, i.e. when solutions set  $x$  of an equation  $\psi_i(x, t) = \dot{\psi}_i(x, \tilde{u}(x), t) = 0$  does not intersect the section  $\Gamma Q \cap \Gamma Q_i$ .

## 5. REALIZATION OF SYNTHESIS ALGORITHM FOR LINEAR SYSTEMS

Let system (1) or (4) to be linear and can be represented by following equations:

$$\dot{x} = Ax + Bu, x(t_0) = x_0, t \geq t_0 \quad (11)$$

$$\dot{x} = Ax + Bu + Dv, x(t_0) = x_0, t \geq t_0 \quad (12)$$

Where matrices A, B, D have formats, concerted with vectors  $x, u, v$  accordingly. And for (12)  $A=A(z), B=B(z), D=D(z)$ .

Besides, the limitation functions  $\psi_i(x, t), i \in \overline{1, \chi}$ , are also linear and are described by correlations

$$\psi_i(x, t) = (\alpha^i, x) - q_i(t) \leq 0, i \in \overline{1, \chi} \quad (13)$$

Where  $\alpha^i$  - given  $n \times 1$ -vectors,  $q_i(t)$  - given scalar and continuously differentiable functions. We are determining synthesized control law in linear form, i.e.  $\tilde{u}(x) = Kx$

Then equations (11), (12) lead to:

$$\dot{x} = \tilde{A}x, x(t_0) = x_0, t \geq t_0 \quad (14)$$

$$\dot{x} = \tilde{A}x + Dv, x(t_0) = x_0, t \geq t_0 \quad (15)$$

Where  $\tilde{A} = A + BK$ . Therefore lets investigate system (14) for a distinctness. If

$$\nabla_x \psi_i = \alpha^i, \quad \frac{\partial \psi_i}{\partial t} = -\dot{q}_i, \quad \text{then } \dot{\psi}_i = (\alpha^i, \tilde{A}x) - \dot{q}_i = (\tilde{A}^T \alpha^i, x) - \dot{q}_i, \quad i \in \overline{1, \chi} \quad (16)$$

Lets use the circumstance, that for an arbitrary hyper plane  $\Gamma \subset \mathbb{R}^n : (\alpha, x) - q = 0$  - vector  $\alpha$  is orthogonal, i.e.  $\alpha \perp \Gamma$

Then considering equations for hyperplanes  $\Gamma Q_i, \Gamma \rho_i$  we will come to

$$\alpha^i \perp \Gamma Q_i, \tilde{A}^T \alpha^i \perp \Gamma \rho_i \quad (17)$$

Lets determine conditions, under which correlations (10) will fulfill.

Lets designate arbitrary apex of polyhedron Q, belonging to a part  $\Gamma Q_i \cap \Gamma Q$ , as  $M_j^i$  i.e.  $M_j^i \in \Gamma Q_i \cap \Gamma Q$

$M_j^i$  is considered to be formed by hyper planes  $\Gamma Q_{ij} \quad i, j \in \overline{1, \chi_j^i}$ . Considering statement 2 for ensuring phase limitations it is enough for each hyper plane  $\Gamma \rho_i$  to intersect an *facet*  $\Gamma Q_i$  in such way, that intersection points do not belong to polyhedron Q. This condition is equal to a statement, that  $M_j^i$  must be in a half-space  $\rho_i^-$ , as it is shown on Fig. 1.

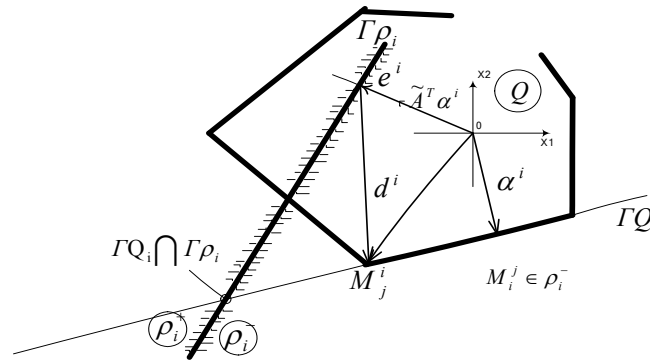


Fig.1 Required arrangement of hyper plane  $\Gamma \rho_i$

Here:  $d^i = M_j^i - e^i$ ;  $e^i$  - vector, which has the same direction as vector  $\tilde{A}^T \alpha^i$  (i.e.  $e^i \perp \Gamma \rho_i$ ), but at the same time its endpoint belongs to  $\Gamma \rho_i$ . Thus, the length of  $e^i$ , i.e.  $\|e^i\|$ , corresponds with the distance between 0 and  $\Gamma \rho_i$  and can be determined from

an equation  $\|e^i\| = \frac{|\dot{q}_i|}{\|\tilde{A}^T \alpha^i\|}$  and for a vector  $e^i$  itself the following correlation is right

$$\|e^i\| = \|e^i\| \frac{\tilde{A}^T \alpha^i}{\|\tilde{A}^T \alpha^i\|} = \frac{\dot{q}_i}{\|\tilde{A}^T \alpha^i\|^2} \cdot \tilde{A}^T \alpha^i \quad (18)$$

Obviously:

$$M_j^i \in \rho_i^-, \quad i, j \in \overline{1, \chi_j^i} \quad (19)$$

if and only if, when the inequality (20) is fulfilled

$$(\tilde{A}^T \alpha^i, d^i) < 0 \quad (20)$$

Considering (16),(18)

$$(\tilde{A}^T \alpha^i, d^i) = (\tilde{A}^T \alpha^i, M_j^i) - \dot{q}_i \quad (21)$$

The result leads to the correctness of the following statements.

### 5.1 Statement 1.

For an arbitrary apex  $M_j^i \in \Gamma Q_i \cap \Gamma Q$ ,  $i_j \in \overline{1, \chi_j^i}$  the condition (19) fulfills if and only if, when the following inequality is right

$$(\tilde{A}^T \alpha^i, M_j^i) - \dot{q}_i < 0, \quad i \in \overline{1, \chi_j^i} \quad (22)$$

### 5.2 Statement 2.

For ensuring the correlation (6) the fulfillment of the following inequalities is needed and enough

$$(\tilde{A}^T \alpha^i, M_j^i) - \dot{q}_i < 0, \quad j \in \overline{1, N_i} \quad (23)$$

where  $N_i$  is total number of apexes on the edge of  $\Gamma Q_i \cap \Gamma Q$

Correlations (23) allow get simple enough solution and analysis for solvability concerted with matrix K, which is linearly in (23) at given phase limitations. Already known methods of solving linear inequalities can be used for this.

Let (23) to fulfill for all hyperplanes  $\Gamma Q_i$ ,  $i_j \in \overline{1, \chi_j^i}$ , forming  $M_j^i$  apex. Then, if Q is a polyhedron, symmetrical to point of origin, then the correlations (23) will be valid for an apex  $\bar{M}_j^i \in Q$  too, where  $\bar{M}_j^i = -M_j^i$  because of the stated symmetry of Q. And  $\bar{M}_j^i$  is forming hyperplanes  $\Gamma \bar{Q}_i$ ,  $i_j \in \overline{1, \chi_j^i}$ , which are symmetrical to  $\Gamma Q_i$ ,  $i_j \in \overline{1, \chi_j^i}$  accordingly.

If the polyhedron Q contains only the edges  $\Gamma Q_i$  and  $\Gamma \bar{Q}_i$ ,  $i_j \in \overline{1, \chi_j^i}$ , which contain  $M_j^i$  and  $\bar{M}_j^i$ , then for synthesizing desired regulator correlations (23) are enough. In case of random phase set it is possible to implement a procedure of its symmetrization.

And then, on the base of (23), in apexes  $M_1, M_2$  we synthesize the regulator.

Let's consider a case, when Q is a rectangular parallelepiped (cuboid), or it can be converted into it.

It is shown, that for symmetric Q (4) can be ensured, if the following equations are fulfilled

$$(\tilde{A}^T \alpha^i, M) - q_i < 0, \quad i \in \overline{1, n} \quad (24),$$

where (formula), M – is an arbitrary apex of polyhedron Q (correlation (24) is fair for any symmetric Q in general case).

Let's designate row vectors of A, B,  $\tilde{A}$  as  $a_i, b_i, \tilde{a}_i$ ,  $i \in \overline{1, n}$ .

Then inequalities (24) will be fulfilled if and only if the following correlations are fulfilled

$$\tilde{a}_i = a_i + k^T b_i \in -K_i \quad i \in \overline{1, n} \quad (25),$$

where  $K_i$  – cone, which is directed along positive (semi)axis  $Ox_i$  and is symmetric to this semi axis.  $K_i$  is defined in the following way.

Let's consider  $M = [m_1 \ m_2 \ \dots \ m_n]^T$ . Then  $x \in K_i$ , if

$$m_i x_i + \sum_{v=1, v \neq i}^n (\pm m_v) x_v \quad (26)$$

where  $(\pm m_v)$  value can be chosen in any combination while  $v \in \overline{1, n} \setminus i$ .

Correlations (25), (26) must be carried to equivalent representation.

Let  $a^i, i \in \overline{1, n}$  to be column vectors of  $A$ -matrix;  $\zeta(B)$  – subspace, which is taut on column vectors of  $B$ -matrix. Then, for equation's (25) solvability the fulfillment of the following correlations is necessary and enough

$$a^i + K_i^* \cap \zeta(B) \neq \emptyset, \quad i \in \overline{1, n} \quad (27)$$

$K_i^*, i \in \overline{1, n}$  – some deformable cones with the same properties as cones  $K_i, i \in \overline{1, n}$ .

## 6. CONCLUSION

The proposed method allows synthesizing a regulator of the desired structure effectively enough. This method is rather simple and the analysis of synthesis problem solvability is also simple, because the system of linear inequalities concerted with the regulator parameters is investigated. If symmetrization of initial phase set is used, then the problem of synthesis may be simplified by reducing its dimension.

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