

## QUASI-HYDRODYNAMIC MODEL AND SMALL SCALE TURBULENCE

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**Abstract.** The averaged quasi-hydrodynamic (QHD) system of equations describing small-scale turbulence and its implementation is presented. The QHD equations differ from the Navier-Stokes ones by additional dissipative terms. For low and middle speed flows these terms are related to the molecular viscosity coefficient. For turbulent flows it must be adjusted to fit the general flow features. A small scale of turbulence has been constructed by using  $k - \varepsilon$  mathematical model for the turbulent kinetic energy. To improve the model in case of plane geometry the problem of vortex shedding past rectangular cylinders with  $Re = 5 \cdot 10^4$  has been considered. The conservative finite-difference schemes on Cartesian and triangular meshes and the algorithms of their numerical implementation have been developed.

The obtained results were compared with results of other authors and physical experiments. For comparison of numerical and experimental results the averaged drag coefficient is used. The results precision of simulation depends on mesh size therefore demands significant computing resources. The general aspect of this paper is increasing quality of numerical solution on rough meshes. This model was tested on a set of simple shapes.

### 1. QHD SYSTEM FOR PULSATING FLOWS

Let us to consider a QHD problem for a case of an incompressible isothermal liquid [1].

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = \frac{\partial}{\partial x} \left( \frac{\tau}{2} \frac{\partial}{\partial x} (\rho u^2 + P) \right) + \frac{\partial}{\partial x} \left( \tau \frac{\partial}{\partial y} (\rho u v) \right) + \frac{\partial}{\partial y} \left( \frac{\tau}{2} \frac{\partial}{\partial y} (\rho v^2 + P) \right) \quad (1)$$

$$\begin{aligned} \frac{\partial \rho u}{\partial t} + \frac{\partial}{\partial x} (\rho u^2 + P) + \frac{\partial}{\partial y} (\rho u v) &= \frac{\partial}{\partial x} \frac{\tau}{2} \frac{\partial}{\partial x} (\rho u^3 + 3 P u) + \\ + \frac{\partial}{\partial x} \tau \frac{\partial}{\partial y} (\rho u^2 v + P v) + \frac{\partial}{\partial y} \frac{\tau}{2} \frac{\partial}{\partial y} (\rho u v^2 + P u) & \end{aligned} \quad (2)$$

$$\begin{aligned} \frac{\partial \rho v}{\partial t} + \frac{\partial}{\partial x} (\rho u v) + \frac{\partial}{\partial y} (\rho v^2 + P) &= \frac{\partial}{\partial x} \frac{\tau}{2} \frac{\partial}{\partial x} (\rho u^2 v + P v) + \\ + \frac{\partial}{\partial x} \tau \frac{\partial}{\partial y} (\rho v^2 u + P u) + \frac{\partial}{\partial y} \frac{\tau}{2} \frac{\partial}{\partial y} (\rho v^3 + 3 P v) & \end{aligned} \quad (3)$$

Here  $\rho$  - density,  $u, v$  - horizontal and vertical velocity components in Cartesian coordinates system,  $P$  - pressure,  $\tau \sim \frac{\mu}{P}$  - molekular collision time,  $\mu$  - viscosity. The difference between dissipative terms of QHD equations and diffusion terms of Naiver-Stokes equations have the asymptotic order of  $O(K_n^{-2})$ , where  $K_n$  is a Knudsen number. But, QHD smooths the solution on a distance with order of molecular free path. This smoothing property has been used for construction of turbulent QHD model. Averaged momentary term of QHD system is written as

$$\overline{\left( \frac{\tau}{2} \frac{\partial}{\partial x} (\rho' u'^2 + P') \right)} = \frac{1}{T} \int_{t-\frac{T}{2}}^{t+\frac{T}{2}} \frac{\partial}{\partial x} \left( \frac{\tau}{2} \frac{\partial}{\partial x} (\rho' u'^2 + P') \right) dt' = \frac{\partial}{\partial x} \left( \frac{\tau}{2} \frac{\partial}{\partial x} (\rho u^2 + P) \right), \quad (4)$$

where  $T$  - time averaging period,  $t$  - time moment,  $u' = u + \Delta u$ ,  $\Delta u$  - pulse component of velocity. Here spatial scale is a step of spatial mesh  $h$ . The turbulent time scale was defined as quantity with order  $\tau_{tur} \sim \frac{k}{\varepsilon}$ , where  $k$  - turbulent kinetic energy( energy of small-scale pulsations),  $\varepsilon$  - speed of thier dissipation. Then, turbulent form of QHD is written as (5), (6):

$$\begin{aligned} \frac{\partial \rho u}{\partial t} + \frac{\partial}{\partial x} (\rho u^2 + P) + \frac{\partial}{\partial y} (\rho u v) &= \frac{\partial}{\partial x} \frac{\tau}{2} \frac{\partial}{\partial x} (\rho u^3 + 3 P u) + \frac{\partial}{\partial x} \tau \frac{\partial}{\partial y} (\rho u^2 v + 3 P v) + \\ &+ \frac{\partial}{\partial y} \frac{\tau}{2} \frac{\partial}{\partial y} (\rho u v^2 + P u) + \frac{\partial}{\partial x} \frac{\tau_{TYP}}{2} \frac{\partial}{\partial x} (3 \rho u \Delta u^2) + \\ &+ \frac{\partial}{\partial x} \tau_{TYP} \frac{\partial}{\partial y} (\rho v \overline{\Delta u^2} + \rho u \overline{\Delta v \Delta u}) + \frac{\partial}{\partial y} \frac{\tau_{TYP}}{2} \frac{\partial}{\partial y} (\rho u \overline{\Delta v^2} + \rho v \overline{\Delta u \Delta v}) \end{aligned} \quad (5)$$

$$\begin{aligned} \frac{\partial \rho v}{\partial t} + \frac{\partial}{\partial x} (\rho u v) + \frac{\partial}{\partial y} (\rho v^2 + P) &= \frac{\partial}{\partial x} \frac{\tau}{2} \frac{\partial}{\partial x} (\rho u^2 v + P v) + \frac{\partial}{\partial x} \tau \frac{\partial}{\partial y} (\rho v^2 u + 3 P u) + \\ &+ \frac{\partial}{\partial y} \frac{\tau}{2} \frac{\partial}{\partial y} (\rho v^3 + 3 P v) + \frac{\partial}{\partial x} \frac{\tau_{TYP}}{2} \frac{\partial}{\partial x} (\rho v \overline{\Delta u^2} + \rho u \overline{\Delta u \Delta v}) + \\ &+ \frac{\partial}{\partial x} \tau_{TYP} \frac{\partial}{\partial y} (\rho u \overline{\Delta v^2} + \rho v \overline{\Delta u \Delta v}) + \frac{\partial}{\partial y} \frac{\tau_{TYP}}{2} \frac{\partial}{\partial y} (3 \rho v \Delta v^2) \end{aligned} \quad (6)$$

Equations (5), (6) describe fluid dynamics parameters as averaged values, i.e.

$$U = \frac{1}{T} \int_{t-\frac{T}{2}}^{t+\frac{T}{2}} u' dt' \quad \overline{\Delta u^2} = \frac{1}{T} \int_{t-\frac{T}{2}}^{t+\frac{T}{2}} (u' - u)^2 dt'$$

We suppose that on small scaled uniform meshes turbulence is homogeneous and isotropic, then:

$$k = \rho \sum_{i,k} \frac{\overline{\Delta u_i \Delta u_k}}{2}, \quad \overline{\rho \Delta u_i \Delta u_k} = 0, \quad k \neq i \quad \Rightarrow \quad \overline{\rho \Delta u_i^2} = \frac{2}{3} K \quad (7)$$

Equations (5), (6) become:

$$\begin{aligned} \frac{\partial \rho u}{\partial t} + \frac{\partial}{\partial x}(\rho u^2 + P) + \frac{\partial}{\partial y}(\rho uv) &= \frac{\partial}{\partial x} \frac{\tau}{2} \frac{\partial}{\partial x}(\rho u^3 + 3Pu) + \frac{\partial}{\partial x} \tau \frac{\partial}{\partial y}(\rho u^2 v + Pv) + \\ &+ \frac{\partial}{\partial y} \frac{\tau}{2} \frac{\partial}{\partial y}(\rho uv^2 + Pu) + \frac{\partial}{\partial x} \mu_T \frac{\partial u}{\partial x} + \frac{\partial}{\partial y} \mu_T \frac{\partial u}{\partial y} + \frac{\partial}{\partial x} 2\mu_T \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right), \end{aligned} \quad (8)$$

$$\begin{aligned} \frac{\partial \rho v}{\partial t} + \frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(\rho v^2 + P) &= \frac{\partial}{\partial x} \frac{\tau}{2} \frac{\partial}{\partial x}(\rho u^2 v + Pv) + \frac{\partial}{\partial x} \tau \frac{\partial}{\partial y}(\rho v^2 u + Pu) + \\ &+ \frac{\partial}{\partial y} \frac{\tau}{2} \frac{\partial}{\partial y}(\rho v^3 + 3Pv) + \frac{\partial}{\partial x} \mu_T \frac{\partial v}{\partial x} + \frac{\partial}{\partial y} \mu_T \frac{\partial v}{\partial y} + \frac{\partial}{\partial y} 2\mu_T \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right), \end{aligned} \quad (9)$$

$$\tau = \frac{2\mu}{P} \quad (10)$$

$$\mu_T = C_\mu \rho \frac{K^2}{\varepsilon} \quad (11)$$

The turbulent viscosity  $\mu_T$  is equal to same value in  $k - \varepsilon$  model. Then  $k - \varepsilon$  system of equations is written as (12),(13):

$$\frac{\partial(\rho K)}{\partial t} + \frac{\partial(\rho u K)}{\partial x} + \frac{\partial(\rho v K)}{\partial y} = \frac{\partial}{\partial x} \left( (\mu + \mu_T) \frac{\partial K}{\partial x} \right) + \frac{\partial}{\partial y} \left( (\mu + \mu_T) \frac{\partial K}{\partial y} \right) + \Omega - \rho \varepsilon \quad (12)$$

$$\frac{\partial(\rho \varepsilon)}{\partial t} + \frac{\partial(\rho u \varepsilon)}{\partial x} + \frac{\partial(\rho v \varepsilon)}{\partial y} = \frac{\partial}{\partial x} \left( (\mu + \mu_T) \frac{\partial \varepsilon}{\partial x} \right) + \frac{\partial}{\partial y} \left( (\mu + \mu_T) \frac{\partial \varepsilon}{\partial y} \right) + C_{\varepsilon_1} \frac{\varepsilon \Omega}{K} - C_{\varepsilon_2} \frac{\rho \varepsilon^2}{K} \quad (13)$$

where  $\Omega = \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial u_j}{\partial x_i}$ ,  $C_\mu, C_{\varepsilon_1}, C_{\varepsilon_2}$  is an empirical constant [3],

$$(C_\mu = 0.09, \quad C_{\varepsilon_1} = 1.44, \quad C_{\varepsilon_2} = 1.92)$$

## 2. NUMERICAL ALGORITHM

The conservative difference schemes for problems (1), (8)-(13) were constructed on uniform rectangular mesh. In calculation area  $D(x,y) \times [0,T]$  we define mesh with space step  $h$  and time step  $h_t$ :

$$\omega_h = \left\{ (x_i, y_j, t_n), \quad x_i = ih, \quad y_j = jh, \quad t_n = nh_t, \quad i = 0..N_x, \quad j = 0..N_y, \quad n = 0..N_T \right\}.$$

Then we take the integral of given equations by time interval  $(0, h_t)$  and control volume  $V_{i,j}$  that belongs to every mesh node. Here every control volume was created by nodes

$$\{p_1, p_2, p_3, p_4\} = \left\{ \left( i - \frac{1}{2}, j - \frac{1}{2} \right); \left( i - \frac{1}{2}, j + \frac{1}{2} \right); \left( i + \frac{1}{2}, j + \frac{1}{2} \right); \left( i + \frac{1}{2}, j - \frac{1}{2} \right) \right\}.$$

For example, represent equation (8) in divergent form and take the integral by time:

$$\frac{1}{h_i} (u^{n+1} - u^n) + \operatorname{div} \begin{pmatrix} \rho u^2 + P \\ \rho uv \end{pmatrix}^n = \tau \operatorname{div} \begin{pmatrix} \frac{1}{2} \frac{\partial}{\partial x} (\rho u^3 + 3Pu) + \frac{\partial}{\partial y} (\rho u^2 v + Pv) \\ \frac{1}{2} \frac{\partial}{\partial y} (\rho uv^2 + Pu) \end{pmatrix}^n + \operatorname{div} \begin{pmatrix} \mu_T \left( 3 \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \\ \mu_T \frac{\partial u}{\partial y} \end{pmatrix}^n \quad (14)$$

Discrete functions  $u, v, P$  inside control volume  $V_{i,j}$  represent corresponding to bilinear interpolation  $f(t_x, t_y, f_1, f_2, f_3, f_4) = (1-t_x) \left[ (1-t_y) f_1 + t_y f_2 \right] + t_x \left[ (1-t_y) f_4 + t_y f_3 \right]$ , where  $f_k$  - values of discrete functions in nodes  $p_k$ ,  $k = 1..4$ ,  $t_x, t_y \in [0, 1]$ . Then take the space integral corresponding to Gauss-Ostrogradsky divergence theorem. The result of this operation is a second order explicit discrete approximation scheme with nine-nodes template. Corresponding to (1) pressure equation is written:

$$\left( \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} \right) = \frac{1}{\tau} \left( \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} \right) - \left( \frac{\partial^2(\rho u^2)}{\partial x^2} + \frac{\partial^2(\rho v^2)}{\partial y^2} + 2 \frac{\partial^2(\rho uv)}{\partial x \partial y} \right) \quad (15)$$

with boundary condition  $\frac{\partial P}{\partial n} = 0$ , where  $n$  is normal to region  $D(x, y)$ .

### 3. SIMULATION RESULTS

To improve the model in case of plane geometry the problem of vortex shedding past rectangular cylinders with different ratios  $\Theta = \frac{\Delta x}{\Delta y}$  and  $Re = 5 \cdot 10^4$  has been considered, where  $\Delta x$  - length,  $\Delta y$  - height of rectangular cylinder measured in number of control volumes. The velocity of cylinder is related to the velocity of fluid which it is  $u = 1$ ,  $v = 0$ . At numerical simulation with  $k - \varepsilon$  model we used two systems of rectangular meshes.

**Table 1.** Mesh parameters. The  $N_x, N_y$  are region sizes measured in number of control volumes.

<b>run1</b> $N_x=1000$ $N_y=250$	$\Delta x$	2	4	6	8	10
	$\Delta y$	10	10	10	10	10
<b>run2</b> $N_x=260$ $N_y=65$	$\Delta x$	1	2	3	4	5
	$\Delta y$	5	5	5	5	5

Figure 1 shows velocities distribution past square cylinder ( $\Theta = 1$ ). The differences between numerical simulation results and physical experiment data [4,5] can be registried with the help of a drag coefficient that is shown on figure 2. Figure 3 shows isolines of pressure distribution for square cylinder. Figure 4 shows the turbulent kinetic energy  $k$  and its dissipation speed  $\varepsilon$  calculated on different meshes past square cylinder. On fine meshes (run2) turbulent kinetic energy  $k$  describes smaller scaled

components of turbulent flow, therefore it decreases and tends to zero when  $h$  tends to zero.

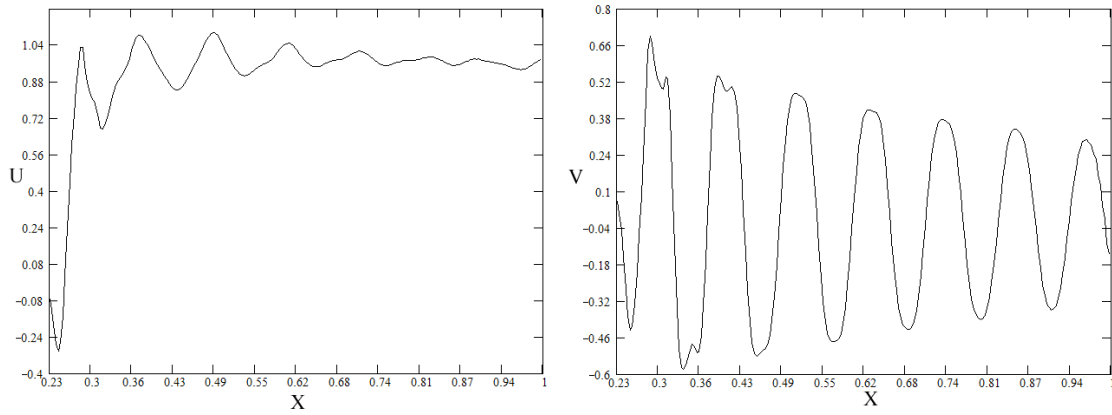


Figure 1. Velocity pulsations.

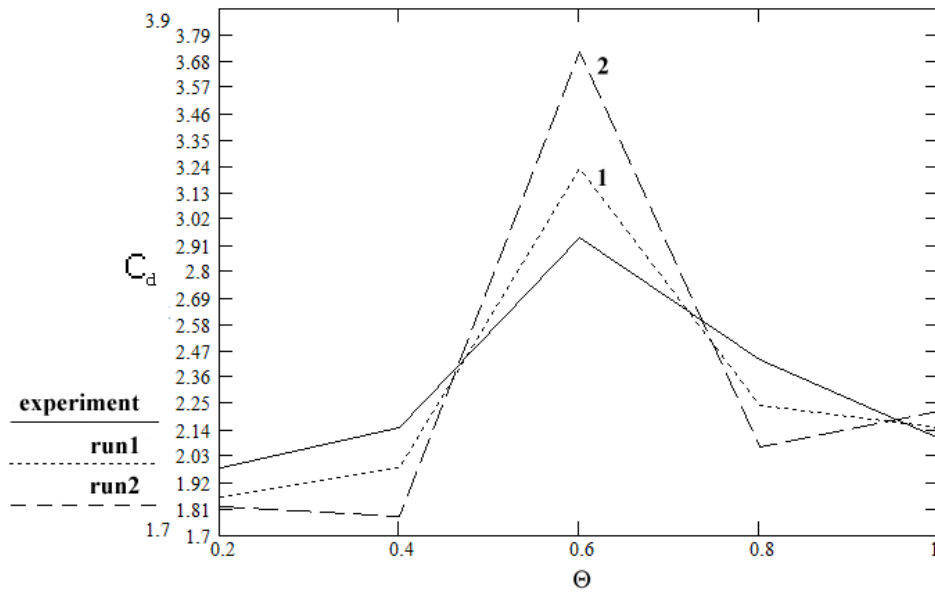


Figure 2. Drag coefficient.

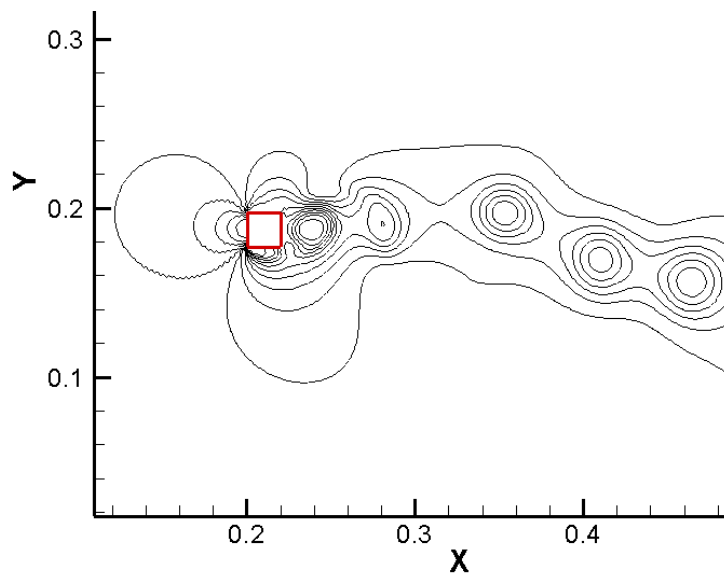
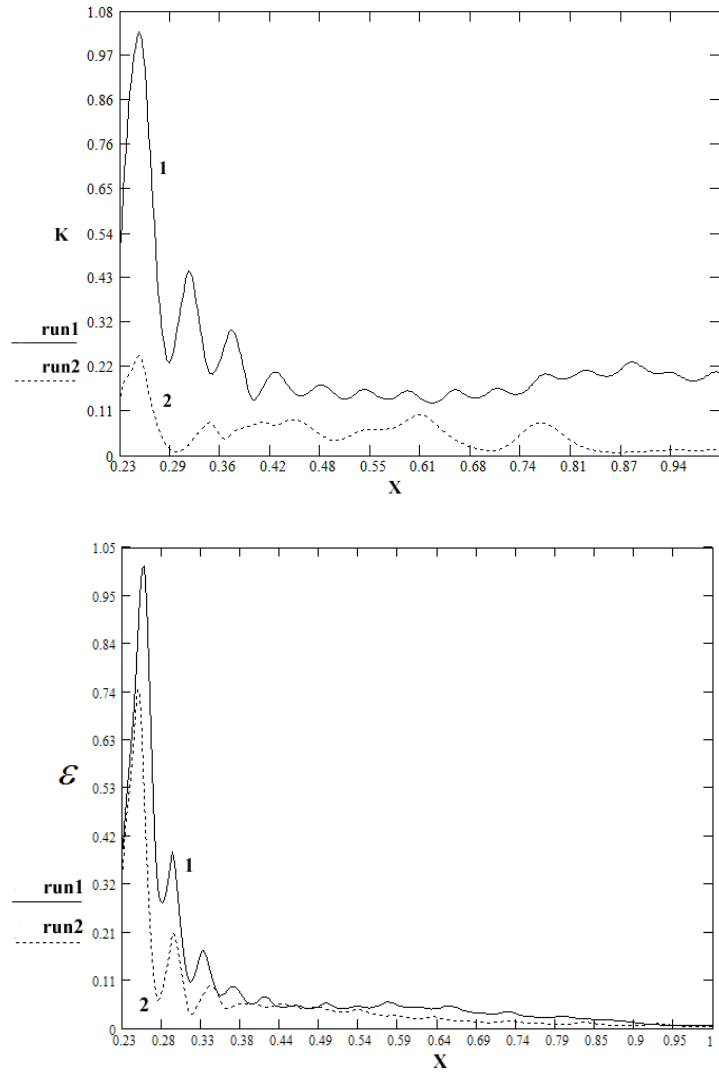


Figure 3. Isolines of pressure.



**Figure 4.** Turbulent kinetic energy  $k$  and their dissipation speed  $\varepsilon$ .

Also we have made direct simulation on triangular unstructured meshes using only equations (1)-(3). Figure 5 shows the triangular mesh is adopted to object geometry. Table 2 represents drag coefficient calculated on triangular mesh with 60000 nodes.

**Table 2.** Drag coefficient.

	$\theta = 0.2$	$\theta = 0.4$	$\theta = 0.6$	$\theta = 0.8$	$\theta = 1.0$
Experiment	1.98	2.15	2.94	2.43	2.1
Numerical simulation	2.03	2.17	2.98	2.46	2.12

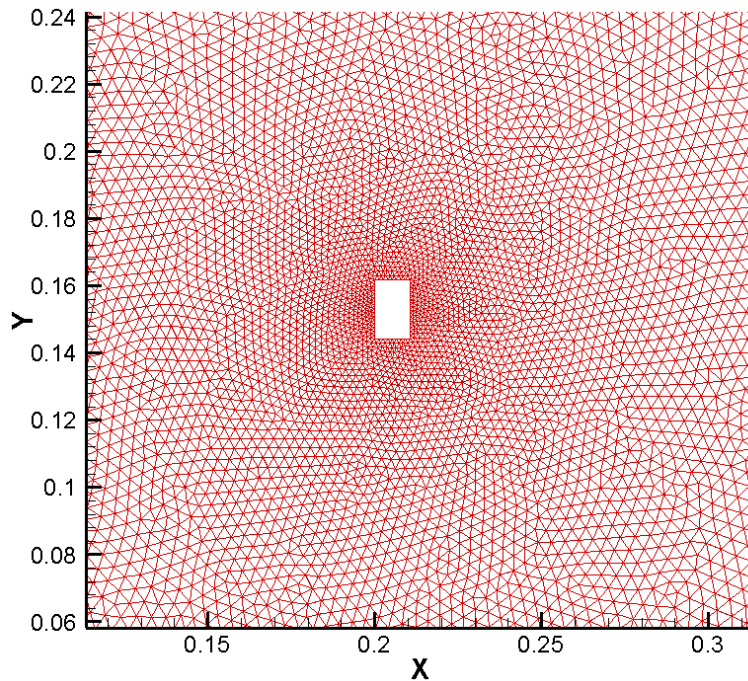


Figure 5. Adaptive triangular mesh.

## 5. REFERENCES

- [1] B.N. Chetverushkin . *Kinetic schemes and system of quasi-hydrodynamics equations* Moscow, Nauka (in Russian) (1980).
- [2] I.V. Abalkin, A.N. Antonov, I.A. Graur, B.N. Chetverushkin. *Use of algebraic model of turbulence for simulation of non-stationary flows in a vicinity of cavities*. *Mathematical modeling*, 14(1), p. 45 (in Russian) (2000).
- [3] O.M. Belocerkovskij, A.M. Oparin, V.M. Chechetkin. *Turbulence New approaches*. Moscow, Nauka, (in Russian)(2002)
- [4] Bearman P.W., Trueman D.M. *An Investigation of the Flow around Rectangular Cylinders*. *Aeronautical Quarterly*, p. 229-237, 1972.
- [5] H. Nakaguchi, K. Hashimoto, S. Muto. *An Experimental study on aerodynamic drag of rectangle cylinders*. *Journal of the Japan Society of Aeronautical and Space Sciences*, Vol 16, p. 1-5, 1968.