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QUASI-HYDRODYNAMIC MODEL AND SMALL SCALE TURBULENCE

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Abstract. The averaged quasi-hydrodynamic (QHD) system of equations describing small-scale turbulence and its implementation is presented. The QHD equations differ from the Navier-Stokes ones by additional dissipative terms. For low and middle speed flows these terms are related to the molecular viscosity coefficient. For turbulent flows it must be adjusted to fit the general flow features. A small scale of turbulence has been constructed by using $k-\varepsilon$ mathematical model for the turbulent kinetic energy. To improve the model in case of plane geometry the problem of vortex shedding past rectangular cylinders with Re= $5\cdot 10^4$ has been considered. The conservative finite-difference schemes on Cartesian and triangular meshes and the algorithms of their numerical implementation have been developed.

The obtained results were compared with results of other authors and physical experiments. For comparison of numerical and experimental results the averaged drag coefficient is used. The results precision of simulation depends on mesh size therefore demands significant computing resources. The general aspect of this paper is increasing quality of numerical solution on rough meshes. This model was tested on a set of simple shapes.

1. QHD SYSTEM FOR PULSATING FLOWS

Let us to consider a QHD problem for a case of an incompressible isothermal liquid [1].

$$\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}=\frac{\partial}{\partial x}\left(\frac{\tau}{2}\frac{\partial}{\partial x}\left(\rho u^2+P\right)\right)+\frac{\partial}{\partial x}\left(\tau\frac{\partial}{\partial y}\left(\rho uv\right)\right)+\frac{\partial}{\partial y}\left(\frac{\tau}{2}\frac{\partial}{\partial y}\left(\rho v^2+P\right)\right) \quad (1)$$

$$\begin{aligned}\frac{\partial \rho u}{\partial t}+\frac{\partial}{\partial x}\left(\rho u^2+P\right)+\frac{\partial}{\partial y}\left(\rho uv\right)=\frac{\partial}{\partial x}\frac{\tau}{2}\frac{\partial}{\partial x}\left(\rho u^3+3Pu\right)+\\ +\frac{\partial}{\partial x}\tau\frac{\partial}{\partial y}\left(\rho u^2v+Pv\right)+\frac{\partial}{\partial y}\frac{\tau}{2}\frac{\partial}{\partial y}\left(\rho uv^2+Pu\right)\end{aligned} \quad (2)$$

$$\begin{aligned}\frac{\partial \rho v}{\partial t}+\frac{\partial}{\partial x}\left(\rho uv\right)+\frac{\partial}{\partial y}\left(\rho v^2+P\right)=\frac{\partial}{\partial x}\frac{\tau}{2}\frac{\partial}{\partial x}\left(\rho u^2v+Pv\right)+\\ +\frac{\partial}{\partial x}\tau\frac{\partial}{\partial y}\left(\rho v^2u+Pu\right)+\frac{\partial}{\partial y}\frac{\tau}{2}\frac{\partial}{\partial y}\left(\rho v^3+3Pv\right)\end{aligned} \quad (3)$$

Here ρ - density, $u\,,v$ - horizontal and vertical velocity components in Cartesian coordinates system , P – pressure, $\tau \sim \dfrac{\mu}{P}$ - molekular collision time, μ - viscosity. The difference between dissipative terms of QHD equations and diffusion terms of Naiver-Stokes equations have the asymptotic order of $O(K_n^2)$, where K_n is a Knudsen number. But, QHD smooths the solution on a distance with order of molecular free path. This smoothing property has been used for construction of turbulent QHD model. Averaged momentary term of QHD system is written as

$$\overline{\frac{\partial}{\partial x}\left(\frac{\tau}{2}\frac{\partial}{\partial x}\left(\rho' u'^2 + P'\right)\right)} = \frac{1}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} \frac{\partial}{\partial x}\left(\frac{\tau}{2}\frac{\partial}{\partial x}\left(\rho' u'^2 + P'\right)\right) dt' = \frac{\partial}{\partial x}\left(\frac{\tau}{2}\frac{\partial}{\partial x}\left(\rho u^2 + P\right)\right), \quad (4)$$

where T - time averaging period, t- time moment, $u' = u + \Delta u$, Δu - pulse component of velocity. Here spatial scale is a step of spatial mesh *h*. The turbulent time scale was defined as quantity with order $\tau_{tur} \sim \dfrac{k}{\varepsilon}$, where k – turbulent kinetic energy(energy of small-scale pulsations), ε -speed of thier dissipation. Then, turbulent form of QHD is written as (5), (6):

$$\frac{\partial \rho u}{\partial t} + \frac{\partial}{\partial x}\left(\rho u^2 + P\right) + \frac{\partial}{\partial y}\left(\rho uv\right) = \frac{\partial}{\partial x}\frac{\tau}{2}\frac{\partial}{\partial x}\left(\rho u^3 + 3Pu\right) + \frac{\partial}{\partial x}\tau\frac{\partial}{\partial y}\left(\rho u^2 v + 3Pv\right) +$$
$$+ \frac{\partial}{\partial y}\frac{\tau}{2}\frac{\partial}{\partial y}\left(\rho uv^2 + Pu\right) + \frac{\partial}{\partial x}\frac{\tau_{TYP}}{2}\frac{\partial}{\partial x}\left(3\rho u \Delta u^2\right) + \quad (5)$$
$$+ \frac{\partial}{\partial x}\tau_{TYP}\frac{\partial}{\partial y}\left(\rho v \overline{\Delta u^2} + \rho u \overline{\Delta v \Delta u}\right) + \frac{\partial}{\partial y}\frac{\tau_{TYP}}{2}\frac{\partial}{\partial y}\left(\rho u \overline{\Delta v^2} + \rho v \overline{\Delta u \Delta v}\right)$$

$$\frac{\partial \rho v}{\partial t} + \frac{\partial}{\partial x}\left(\rho uv\right) + \frac{\partial}{\partial y}\left(\rho v^2 + P\right) = \frac{\partial}{\partial x}\frac{\tau}{2}\frac{\partial}{\partial x}\left(\rho u^2 v + Pv\right) + \frac{\partial}{\partial x}\tau\frac{\partial}{\partial y}\left(\rho v^2 u + 3Pu\right) +$$
$$+ \frac{\partial}{\partial y}\frac{\tau}{2}\frac{\partial}{\partial y}\left(\rho v^3 + 3Pv\right) + \frac{\partial}{\partial x}\frac{\tau_{TYP}}{2}\frac{\partial}{\partial x}\left(\rho v \overline{\Delta u^2} + \rho u \overline{\Delta u \Delta v}\right) + \quad (6)$$
$$+ \frac{\partial}{\partial x}\tau_{TYP}\frac{\partial}{\partial y}\left(\rho u \overline{\Delta v^2} + \rho v \overline{\Delta u \Delta v}\right) + \frac{\partial}{\partial y}\frac{\tau_{TYP}}{2}\frac{\partial}{\partial y}\left(3\rho v \Delta v^2\right)$$

Equations (5), (6) describe fluid dynamics parameters as averaged values, i.e.

$$U = \frac{1}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} u' dt' \qquad \overline{\Delta u^2} = \frac{1}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} (u' - u)^2 dt'$$

We suppose that on small scaled uniform meshes turbulence is homogeneous and isotropic, then:

$$k = \rho \sum_{i,k} \frac{\overline{\Delta u_i \Delta u_k}}{2}, \quad \rho\, \overline{\Delta u_i \Delta u_k} = 0, \quad k \neq i \quad \Rightarrow \quad \rho \overline{\Delta u_i^2} = \frac{2}{3}K \quad (7)$$

Equations (5), (6) become:

$$\frac{\partial \rho u}{\partial t}+\frac{\partial}{\partial x}\left(\rho u^2+P\right)+\frac{\partial}{\partial y}\left(\rho uv\right)=\frac{\partial}{\partial x}\frac{\tau}{2}\frac{\partial}{\partial x}\left(\rho u^3+3Pu\right)+\frac{\partial}{\partial x}\tau\frac{\partial}{\partial y}\left(\rho u^2 v+Pv\right)+$$
$$+\frac{\partial}{\partial y}\frac{\tau}{2}\frac{\partial}{\partial y}\left(\rho uv^2+Pu\right)+\frac{\partial}{\partial x}\mu_T\frac{\partial u}{\partial x}+\frac{\partial}{\partial y}\mu_T\frac{\partial u}{\partial y}+\frac{\partial}{\partial x}2\mu_T\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right), \quad (8)$$

$$\frac{\partial \rho v}{\partial t}+\frac{\partial}{\partial x}\left(\rho uv\right)+\frac{\partial}{\partial y}\left(\rho v^2+P\right)=\frac{\partial}{\partial x}\frac{\tau}{2}\frac{\partial}{\partial x}\left(\rho u^2 v+Pv\right)+\frac{\partial}{\partial x}\tau\frac{\partial}{\partial y}\left(\rho v^2 u+Pu\right)+$$
$$+\frac{\partial}{\partial y}\frac{\tau}{2}\frac{\partial}{\partial y}\left(\rho v^3+3Pv\right)+\frac{\partial}{\partial x}\mu_T\frac{\partial v}{\partial x}+\frac{\partial}{\partial y}\mu_T\frac{\partial v}{\partial y}+\frac{\partial}{\partial y}2\mu_T\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right), \quad (9)$$

$$\tau=\frac{2\mu}{P} \quad (10)$$

$$\mu_T=C_\mu\rho\frac{K^2}{\varepsilon} \quad (11)$$

The turbulent viscosity μ_T is equal to same value in $k-\varepsilon$ model. Then $k-\varepsilon$ system of equations is written as (12),(13):

$$\frac{\partial(\rho K)}{\partial t}+\frac{\partial(\rho uK)}{\partial x}+\frac{\partial(\rho vK)}{\partial y}=\frac{\partial}{\partial x}\left(\left(\mu+\mu_T\right)\frac{\partial K}{\partial x}\right)+\frac{\partial}{\partial y}\left(\left(\mu+\mu_T\right)\frac{\partial K}{\partial y}\right)+\Omega-\rho\varepsilon \quad (12)$$

$$\frac{\partial(\rho\varepsilon)}{\partial t}+\frac{\partial(\rho u\varepsilon)}{\partial x}+\frac{\partial(\rho v\varepsilon)}{\partial y}=\frac{\partial}{\partial x}\left(\left(\mu+\mu_T\right)\frac{\partial \varepsilon}{\partial x}\right)+\frac{\partial}{\partial y}\left(\left(\mu+\mu_T\right)\frac{\partial \varepsilon}{\partial y}\right)+C_{\varepsilon_1}\frac{\varepsilon\Omega}{K}-C_{\varepsilon_2}\frac{\rho\varepsilon^2}{K} \quad (13)$$

where $\Omega=\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\frac{\partial u_j}{\partial x_i}$, $C_\mu, C_{\varepsilon_1}, C_{\varepsilon_2}$ is an empirical constant [3], ($C_\mu=0.09$, $C_{\varepsilon_1}=1.44$, $C_{\varepsilon_2}=1.92$)

2. NUMERICAL ALGORITHM

The conservative difference schemes for problems (1), (8)-(13) were constructed on uniform rectangular mesh. In calculation area $D(x,y)\times[0,T]$ we define mesh with space step h and time step h_t :

$\omega_h=\left\{\left(x_i,y_j,t_n\right),\ \ x_i=ih,\ \ y_j=jh,\ \ t_n=nh_t,\ \ i=0..N_x,\ \ j=0..N_y,\ \ n=0..N_T\right\}$.

Then we take the integral of given equations by time interval $\left(0,h_t\right)$ and control volume $V_{i,j}$ that belongs to every mesh node. Here every control volume was created by nodes

$\left\{p_1,p_2,p_3,p_4\right\}=\left\{\left(i-\frac{1}{2},j-\frac{1}{2}\right);\left(i-\frac{1}{2},j+\frac{1}{2}\right);\left(i+\frac{1}{2},j+\frac{1}{2}\right);\left(i+\frac{1}{2},j-\frac{1}{2}\right)\right\}$.

For example, represent equation (8) in divergent form and take the integral by time:

$$\frac{1}{h_t}\left(u^{n+1}-u^n\right)+div\begin{pmatrix}\rho u^2+P\\ \rho uv\end{pmatrix}^n=$$
$$\tau div\begin{pmatrix}\frac{1}{2}\frac{\partial}{\partial x}\left(\rho u^3+3Pu\right)+\frac{\partial}{\partial y}\left(\rho u^2 v+Pv\right)\\ \frac{1}{2}\frac{\partial}{\partial y}\left(\rho uv^2+Pu\right)\end{pmatrix}^n+div\begin{pmatrix}\mu_T\left(3\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\\ \mu_T\frac{\partial u}{\partial y}\end{pmatrix}^n \quad (14)$$

Discrete functions u, v, P inside control volume $V_{i,j}$ represent corresponding to bilinear interpolation $f(t_x, t_y, f_1, f_2, f_3, f_4) = (1-t_x)\left[(1-t_y) f_1 + t_y f_2\right] + t_x\left[(1-t_y) f_4 + t_y f_3\right]$, where f_k - values of discrete functions in nodes p_k, $k = 1..4$, $t_x, t_y \in [0,1]$. Then take the space integral corresponding to Gauss-Ostrogradsky divergence theorem. The result of this operation is a second order explicit discrete approximation scheme with nine-nodes template. Corresponding to (1) pressure equation is written:

$$\left(\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}\right)=\frac{1}{\tau}\left(\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}\right)-\left(\frac{\partial^2\left(\rho u^2\right)}{\partial x^2}+\frac{\partial^2\left(\rho v^2\right)}{\partial y^2}+2\frac{\partial^2(\rho uv)}{\partial x\partial y}\right) \quad (15)$$

with boundary condition $\frac{\partial P}{\partial n} = 0$, where *n* is normal to region $D(x, y)$.

3. SIMULATION RESULTS

To improve the model in case of plane geometry the problem of vortex shedding past rectangular cylinders with different ratios $\Theta = \frac{\Delta x}{\Delta y}$ and $Re = 5\cdot 10^4$ has been considered, where Δx - length, Δy - height of rectangular cylinder measured in number of control volumes. The velocity of cylinder is relatived to the velocity of fluid which it is $u = 1$, *v=0.* At numerical simulation with $k-\varepsilon$ model we used two systems of rectangular meshes.

Table1. Mesh parameters. The Nx, Ny are region sizes measured in number of control volumes.

run1 Nx=1000 Ny=250	Δx	2	4	6	8	10
	Δy	10	10	10	10	10
run2 Nx=260 Ny=65	Δx	1	2	3	4	5
	Δy	5	5	5	5	5

Figure 1 shows velocities destribution past square cylinder ($\Theta = 1$). The differences between numerical simulation results and physical experiment data [4,5] can be registried with the help of a drag coefficient that is shown on figure 2. Figure 3 shows isolines of pressure distribution for square cylinder. Figure 4 shows the turbulent kinetic energy k and its dissipation speed ε calculated on different meshes past square cylinder. On fine meshes (run2) tulbulent kinetic energy *k* describes smaller scaled components of turbulent flow, therefore it decreases and tends to zero when *h* tends to zero.

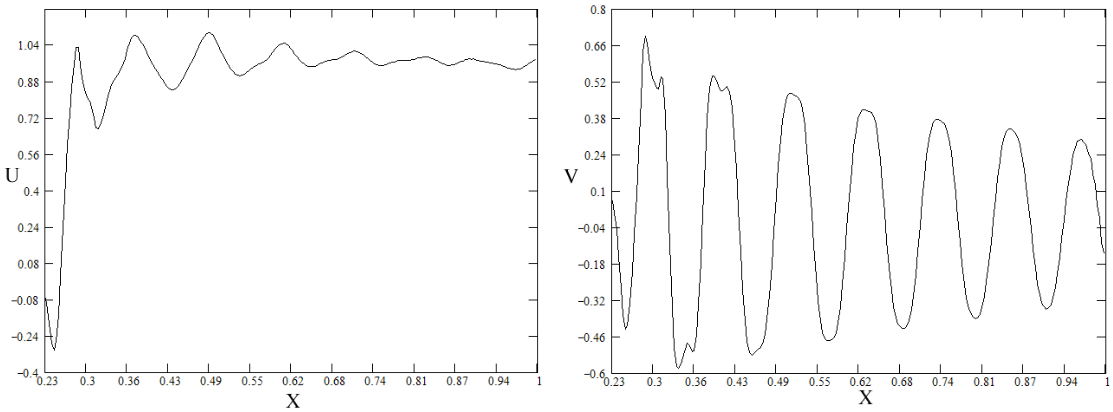

Figure 1. Velosity pulsations.

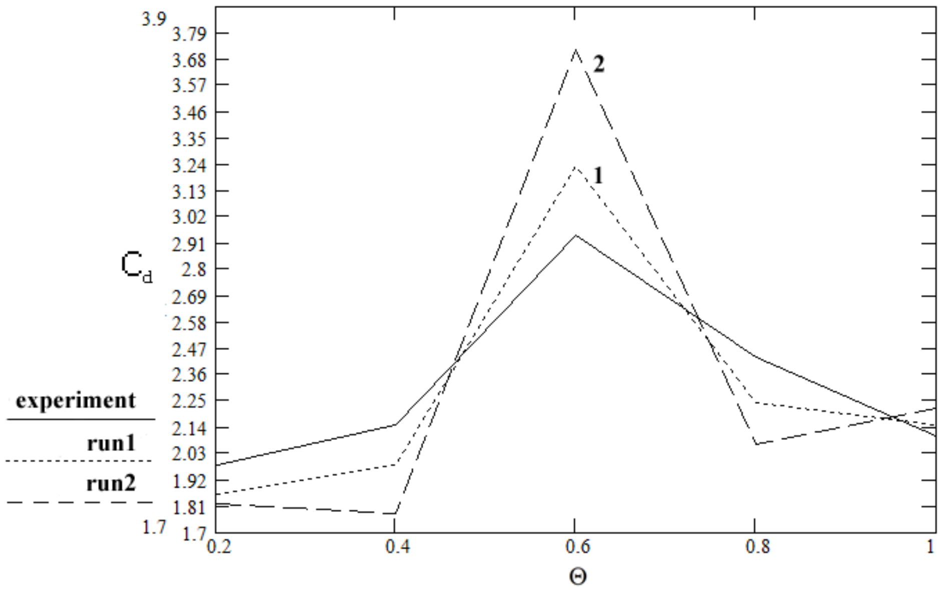

Figure 2. Drag coefficient.

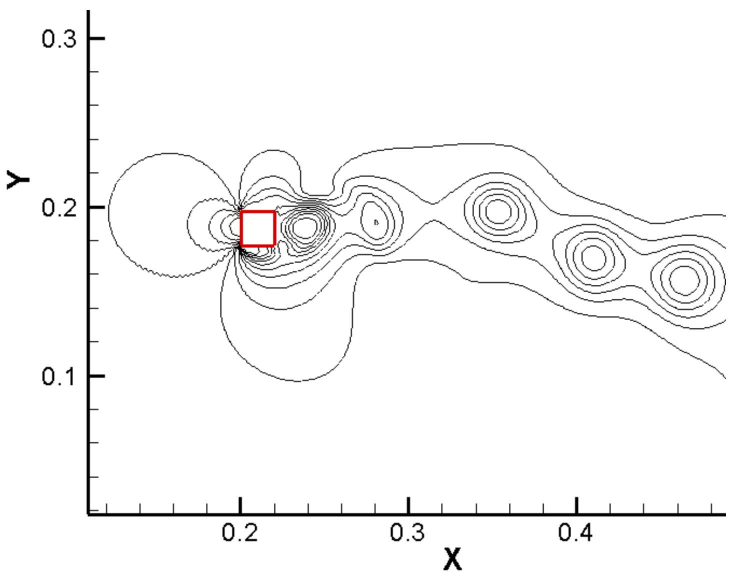

Figure 3. Isolines of pressure.

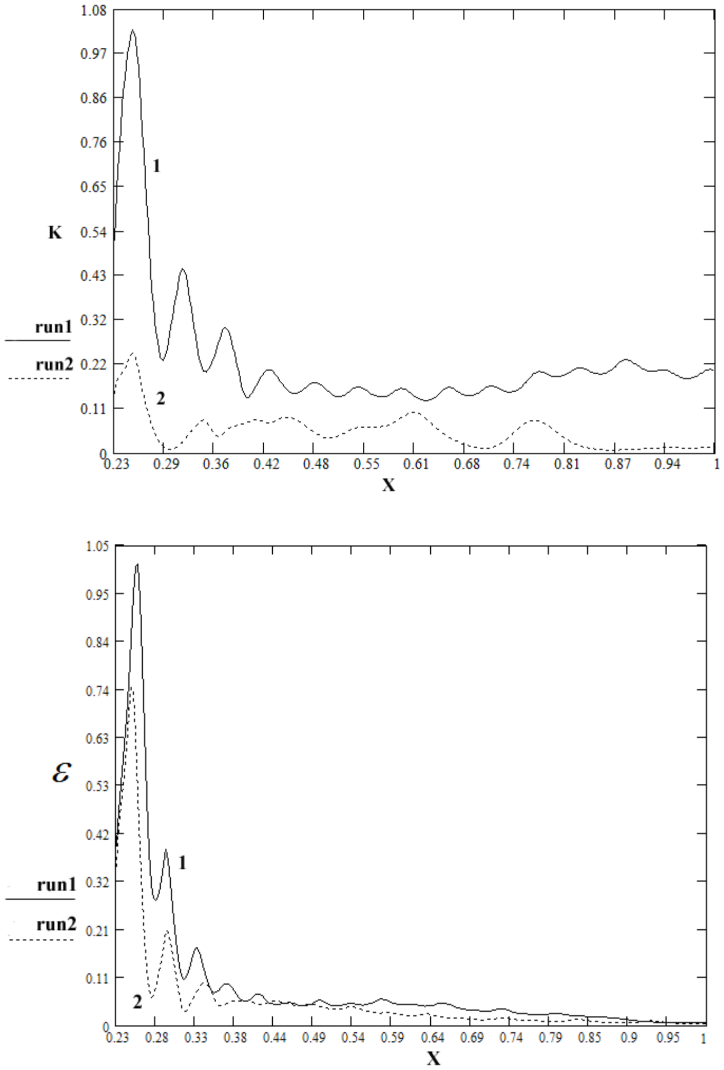

Figure 4. Turbulent kinetic energy k and their dissipation speed ε.

Also we have made direct simulation on triangular unstructered meshes using only equations (1)-(3). Figure 5 shows the triangular mesh is adopted to object geometry. Table 2 represents drag coefficient calculated on triangular mesh with 60000 nodes.

Table 2. Drag coefficient.

	$\theta = 0.2$	$\theta = 0.4$	$\theta = 0.6$	$\theta = 0.8$	$\theta = 1.0$
Experiment	1.98	2.15	2.94	2.43	2.1
Numerical simulation	2.03	2.17	2.98	2.46	2.12

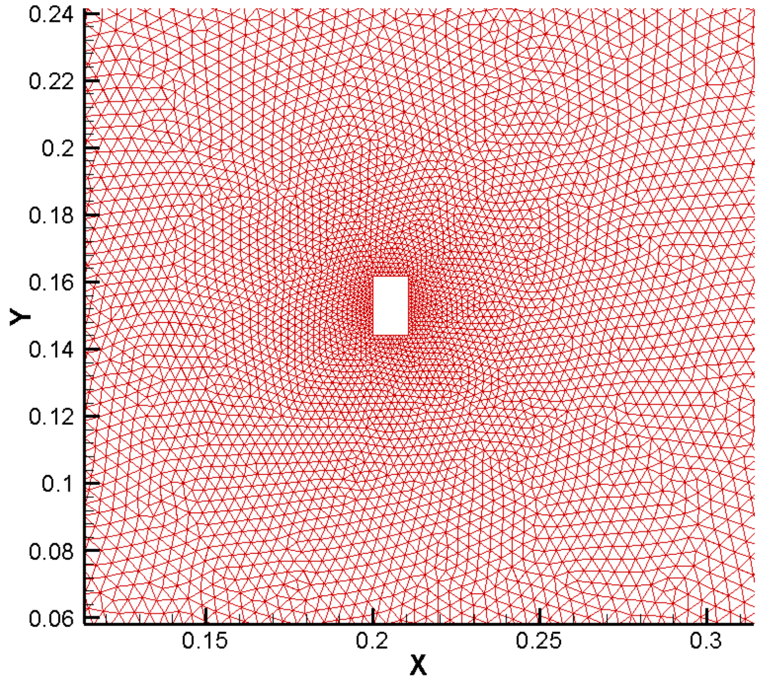

Figure 5. Adaptive triangular mesh.

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