EXACTLY SOLVABLE MODELS OF NONSTATIONARY TURBULENCE IN BOSE CONDENSATE

Pavel A. Kuzmin

Department of Physics, Moscow State University, Moscow 119992, Russia

Key words: quantum turbulence, Bose condensate, nonstationary decay

Abstract. In this note we study a turbulence decay mechanism in the superfluid liquid. We proceed with developement of master equation approach introduced Copeland, Kibble, Steer and Nemirovskii. We obtain the full rate of reconnection in presence of normal component. We also discuss different random-walk models of vortex filaments. We obtain the expression for the reconnection rate in the nonstationary vortex tangle for these models. The equation for the full number of vortex loops is derived. We also obtain the expression for the relaxation time.

1. INTRODUCTION

The description of the mechanism of decaying the turbulent state of a superfluid liquid is an important problem of hydrodynamics.[13] For temperatures $T > 1K$, the viscosity of the normal component and mutual friction are the main dissipation factors. For lower temperatures, these factors are negligible, because the density of the normal component is low. A valuable progress in the theory of superfluid turbulence has been made in recent years. Significant results on the regimes and spectra of turbulence were obtained by Volovik[14] and others[6, 5, 2, 12].

Superfluid flows are described mathematically by the Landau two-fluid model, in contrast to classical ideal or viscous fluids, which are described by the Euler or Navier-Stokes equations, respectively. When the temperature is low enough for the normal fluid to be negligible, Landau's model reduces to the Euler equation for an ideal fluid, which is irrotational except on singular vortex lines around which the circulation of the velocity is quantized. The quantum nature of velocity circulation appears, in this model, as a supplementary condition, compatible with the Euler equation. When both normal fluid and superfluid vortices are present, their interaction, called "mutual friction," must be taken into account. Such models are necessary, for example, to study superfluid turbulence in the counterflow produced by a heat current. At low temperatures, an alternative mathematical description of superflows is given by the Gross-Pitaevskii equation. The GP is a partial differential equation for a complex wave field related to the superflow's density and velocity by Madelung's transformation. The superflow is irrotational, except near the nodal lines (vortex lines) of the complex wave field. These lines are known to follow Eulerian dynamics. These topological defects correspond to the quantum vortices of superfluid helium; they appear naturally in this model. In this context, GP is the correct dynamical equation of motion for superfluids. GP has been shown to contain intricate dynamical mechanisms, such as vortex reconnection, vortex nucleation, and vortex-sound interaction. However, there are no exact solutions in this model. Thus, it is interesting to study any exactly solvable models.

This paper is devoted to the further developement the rate equation approach to the superfluid turbulence. In this work, the evolution of a chaotic uniform isotropic vortex tangle in infinite space is analyzed. We do not assume temperature to be low: we take into account the density of the normal component and the mutual friction. The main results are equation (13) for the reconnection rate, and equations () () for the full number of vortex filanents and reconnection rate in substantially nonstationary vortex tangle. The analysis of the evolution of the vortex tangle is based on the master equation for the length distribution of vortex loops. This method was developed in [7, 8] for cosmic strings and was recently used in [9] to obtain the stationary length distribution of vortex loops in superfluid helium. In the framework of this approach, Nemirovskii [9] also derived a relation between the mean radius of curvature of a filament and the mean distance between filaments in the stationary vortex tangle, the Vinen equation for the decay of turbulence in the quasi-stationary regime owing to the Kelvin waves, and calculated the reconnection frequency for the stationary state. In[3], the limit of applicability of the model and the relationship between results of[10, 7, 3] and the modern hydrodynamic theory of[14, 6, 5, 2, 12] were discussed.

2. A MASTER EQUATION APPROACH

In this section we review some previously published results. Let us recall the main concepts and results of [10, 7, 3, 4]. Turbulence is considered as the chaotic tangle of vortex filaments interacting when they intersect each other. The state at a time t is specified by the length distribution density $n(l,t)$ of open vortex linex or closed vortex loops. There are two types of reconnection processes, the fusion of two open filaments (closed loops) into a single filament (loop) and the breakdown of the filanent (loop) into two filaments (loops). These processes are accompanied by the emission of phonons and rarefaction pulses. The probability $A(l_1,l_2,l)$ of fusion of two lines (loops) with lengths l_1 and l_2 and the probability of breakdown $B(l_1,l_2,l)$ of the line (loop) with length l into two lines (loops) with lengths l_1 and l_2 were calculated in[8]. Calculations assume the open line to be Brownian random walk and give the expressions for $A(l_1,l_2,l)$ and $B(l_1,l_2,l)$:

$$A(l_1, l_2, l) = b_m V_l l_1 l_2, \\ B(l_1, l_2, l) = b_s V_l l (\xi_0 l_1)^{-\frac{3}{2}}. \tag{1}$$

Here b_m, b_s are certain constants and V_l is a characteristic velocity of vortex filaments, which is related to the filament radius of curvature ξ as $V_l = \kappa/\xi$. Let ξ_0 be the persistency length of Wiener distriburion of random walk. Suppose ξ_0 be of the order of the mean radius of curvature of vortex filaments. Nemirovskii[10] obtained the following stationary solution

$$n_0(l) = C_{\text{VLD}} \xi_0^{-\frac{3}{2}} l^{-\frac{5}{2}} \tag{2}$$

of the kinetic equation

$$\frac{\partial n(l,t)}{\partial t} = I_0\left[n(l,t)\right] \quad (3)$$

with the collision integral

$$\begin{aligned} I_0\left[n(l,t)\right] = &\int A(l_1,l_2,l)n(l_1,t)n(l_2,t)\delta(l-l_1-l_2)dl_1dl_2 - \int A(l_1,l,l_2)n(l,t)n(l_1,t)\delta(l_2-l_1-l)dl_1dl_2 \\ &- \int A(l_2,l,l_1)n(l,t)n(l_2,t)\delta(l_1-l_2-l)dl_1dl_2 - \int B(l_1,l_2,l)n(l,t)\delta(l-l_1-l_2)dl_1dl_2 \\ &+ \int B(l,l_2,l_1)n(l_1,t)\delta(l_1-l_2-l)dl_1dl_2 + \int B(l,l_1,l_2)n(l_2,t)\delta(l_2-l_1-l)dl_1dl_2. \end{aligned} \quad (4)$$

Here C_{VLD} denotes a constant of the order of unity.Therefore the total length of vortex filaments has the form

$$\mathcal{L} = \int l n(l)dl = 2C_{\mathrm{VLD}}/\xi_0^2$$

Nemirovskii also derived the Vinen equation, obtained the full rate of reconnection in the stationary state (2). A slightly modified collision integral with the inclusion of energy dissipation upon reconnection was studied by the author,[3] where he obtained the law of nonstationary decay of the dense vortex tangle:

$$\frac{d\mathcal{L}}{dt} = -\Delta l b_m V_l \mathcal{L}^2(t) - 2b_s V_l \mathcal{L}(t)\frac{1}{\xi_0}\left[2\frac{\Delta l/\xi_0 + 1}{\sqrt{1+\Delta l/2\xi_0}} - 2\right], \quad (6)$$

and estimated the characteristic times of the decay. The author also derived the expression for the full rate of reconnection in the nonstationary state:

$$\dot{N}_{rec} = b_m V_l \mathcal{L}^2 + 2b_s V_l \xi_0^{-2}\mathcal{L}. \quad (7)$$

3. RECONNECTION RATE IN THE PRESENCE OF NORMAL COMPONENT

In Ref. [7] Nemirovskii dervied the full Vinen equation with inclusion of the term which describes the growth of the vortex tangle due to mutual friction. It is worth nothing that one can proceed with the similar calculations to obtain the reconnection rate in the presence of mutual friction:

$$\dot{N} = \int \frac{\partial n(l,t)}{\partial t} dl = -\int \frac{\partial n(l,t)}{\partial l} \frac{\partial l}{\partial t} dl =$$
$$- \left[\frac{\alpha I_l V_{ns}}{\sqrt{2} c_2 \xi_0} - \frac{\alpha \beta}{2\xi_0^2} \right] \int \frac{\partial n(l,t)}{\partial l} l dl = \tag{8}$$
$$\alpha \frac{I_l V_{ns}}{\sqrt{2} c_2 \xi_0} N(t) - \frac{\alpha \beta}{2\xi_0^2} N(t),$$

where $N(t) = \int n(l,t)dl$ is the full number of vortex loops, I_l, α, β, c_2 are temperature dependent structure constants introduced in[11, 9]. Taking into account that the relations

$$N = \frac{1}{6c_2^2\xi_0^3} \text{ and } \mathcal{L} = \frac{1}{2c_2^2\xi_0^2}$$

are fulfilled, we obtain the final equation for the reconnection rate in the stationary state:

$$\dot{N}_{rec} = \frac{\sqrt{2}\alpha I_l V_{ns} c_2}{9} \mathcal{L}^2 + \frac{\sqrt{2}\alpha\beta c_2^3}{9} \mathcal{L}^{5/2}. \tag{9}$$

Here we assumed that $\mathcal{L}(t)$ changes much slower than the vortex tangle relaxes to the equilibrium distribution. The relaxation time estimation is given by the formula (17) below. If the self-induced force is dominant, the first term is negligible and one has $\dot{N}_{rec} \sim \mathcal{L}^{5/2}$. On the contrary, if the counterflow force prevails, we obtain the law $\dot{N}_{rec} \sim \mathcal{L}^2$. Both results were obtained in Ref.[1] using a simple qualitative arguments. Equation (9) obtained describes the crossover between the two regimes mentioned in Ref. [1].

4. THE MODEL WITH CLOSED LOOPS

One can derive the equation for the full number of vortex loops. Let us integrate equation (3) with collision integral I_0 with respect to l:

$$\dot{N}(t) = \int \frac{\partial n(l,t)}{\partial t} dl = -b_m V_l \mathcal{L}^2(t) + 2b_s V_l \xi_0^{-2} \mathcal{L}(t). \tag{10}$$

It should be noted that the constant C in the powerlike solution $n(l) = Cl^{-5/2}$, as well as the relation (5) can be derived in a more straightforward way. Indeed, for the equilibrium state we have $dN/dt = 0$. Therefore one has $\mathcal{L} = 2\frac{b_s}{b_m}\frac{1}{\xi_0^2}$ and simple computations yield $C = \frac{b_s}{b_m}\frac{1}{\xi_0^{3/2}}$.

However, there is an obvious problem with equation (10). Indeed, under the assumption of slow change of $\mathcal{L}(t)$ equation (10) implies that the quantity $N(t)$ grows or decreases to the infinity for any state $n(l)$, excepting the stationary one (or any other state that obeys (5)). This inconsistency origins from the roughness of the model of random walk selected. There is another model for isotropic Brownian random walk derived in Ref. [8]. This model, unlike (1), takes into account the closeness of the vortex loops and seems to be more relevant. The expressions for $\tilde{A}(l_1,l_2,l)$ and $\tilde{B}(l_1,l_2,l)$ in this model have the form:[8]

$$
\begin{aligned}
\tilde{A}(l_1, l_2, l) &= b_m V_l l_1 l_2, \\
\tilde{B}(l_1, l_2, l) &= \tilde{b}_s V_l \frac{l^{5/2}}{\xi_0^{3/2} l_1^{3/2} l_2^{3/2}},
\end{aligned}
\tag{11}
$$

where ξ_0 may be identified with the mean radius of curvature of vortex filaments without any additional assumptions.

One can overcome the inconsistency noticed using the rates (11) instead of (1). Let us substitute (11) into the collision integral:

$$
\begin{aligned}
I_1\left[n(l,t)\right] = &\int b_m V_l l_1 l_2 n(l_1,t) n(l_2,t) \delta(l - l_1 - l_2) dl_1 dl_2 - \int b_m V_l l_1 l n(l,t) n(l_1, \\
&- \int b_m V_l l_2 l n(l,t) n(l_2,t) \delta(l_1 - l_2 - l) dl_1 dl_2 - \int \tilde{b}_s V_l \xi_0^{-3/2} l_1^{-3/2} l_2^{-3/2} l^{5/} \\
&+ \int \tilde{b}_s V_l \xi_0^{-3/2} l^{-3/2} l_2^{-3/2} l_1^{5/2} n(l_1,t) \delta(l_1 - l_2 - l) dl_1 dl_2 + \int \tilde{b}_s V_l \xi_0^{-3/2} l^{-3}
\end{aligned}
\tag{12}
$$

Now, let us integrate the last equation with respect to l:

$$
\begin{aligned}
\dot{N}(t) = &\int \frac{\partial n(l,t)}{\partial t} = -b_m V_l \mathcal{L}^2(t) \\
&+ \int \left[\tilde{b}_s V_l \xi_0^{-3/2} l^{5/2} \int\limits_{\xi_0}^{l-\xi_0} l_1^{-3/2} (l - l_1)^{-3/2} dl_1 \right] n(l,t) dl \\
= &- b_m V_l \mathcal{L}^2(t) \\
&+ \int\limits_{2\xi_0}^{\infty} \left[\tilde{b}_s V_l \xi_0^{-2} l^{1/2} (l - \xi_0)^{-1/2} \left(4l - 8\xi_0\right) \right] n(l,t) dl.
\end{aligned}
\tag{13}
$$

The function $\sqrt{\frac{l}{l-\xi_0}}$ varies from $\sqrt{2}$ to 1 on the the range of integration $[2\xi_0, \infty)$, therefore one can replace it with a constant, say $C_1 \in (1, \sqrt{2}]$. This gives us the final equation for the full number of loops:

$$
\begin{aligned}
\dot{N}(t) &= - b_m V_l \mathcal{L}^2(t) + 4\tilde{b}_s V_l \xi_0^{-2} C_1 \left(\mathcal{L}(t) - 2\xi_0 N(t)\right) \\
&= -b_m V_l \mathcal{L}^2(t) + 4\tilde{b}_s V_l \xi_0^{-2} C_1 \left(\langle l \rangle - 2\xi_0\right) N(t),
\end{aligned}
\tag{14}
$$

where $\langle l \rangle = \mathcal{L}(t)/N(t)$ denotes the mean length of the vortex loop. It is worth nothing that $\langle l \rangle - 2\xi_0 \geq 0$ and the last term of (14) is necessarily positive. Thus the full rate of reconnection

$$\dot{N}_{rec} = b_m V_l \mathcal{L}^2(t) + 4\tilde{b}_s V_l \xi_0^{-2} C_1 \left(\langle l \rangle - 2\xi_0 \right) N(t)$$

is positive.

Assume again than $\mathcal{L}(t)$ changes much slower than $N(t)$, i.e. assume the dissipation small. Thus the first line of equation (14) reads as an evolution equation for $N(t)$ whose solution has the form

$$N(t) = \left(N(0) - N_{st} \right) e^{-\frac{t}{T}} + N_{st}, \tag{15}$$

where

$$N_{st} = \frac{\mathcal{L}(t)}{2\xi_0} - \frac{b_m}{8b_s C_1} \xi_0 \mathcal{L}^2(t) \tag{16}$$

denotes the stationary number of vortex loops and

$$T = \frac{\xi_0}{8C_1 b_s V_l} = \frac{\xi_0^2}{8C_1 b_s \kappa} \tag{17}$$

denotes the relaxation time for model (11). One can regard relation (16) as a counterpart to (5) for the random walk model with closed filaments. It relates the full vortex line density to the mean radius of curvature of vortex filaments in the (unknown) stationary state:

$$\mathcal{L} = \frac{8b_s C_1}{b_m} \frac{1}{\xi_0^2} \left[\frac{1}{2} - \frac{\xi_0}{\langle l \rangle} \right].$$

Since the exprission in the square brackets is positive, we have $\mathcal{L} \geq 0$.

5. REFERENCES

[1] C. F. Barenghi and D. C. Samuels, J. Low Temp. Phys. 136 (2004), 281.

[2] A. P. Finne, V. B. Eltsov, R. Hanninen, et al., cond-mat/0606619.

[3] P. A. Kuz'min, JETP Lett. 84 (2006), 204.

[4] P. A Kuzmin, Phys. Lett. A 362 (2006), 84

[5] V. S. L'vov, S. V. Nazarenko, and L. Skrbek, nlin.CD/0606002.

[6] V. S. L'vov, S. V. Nazarenko, and G. E. Volovik, JETP Lett. 80 (2004), 479.

[7] S. K. Nemirovskii, cond-mat/0610234.

[8] S. K. Nemirovskii, cond-mat/0505441.

[9] S. K. Nemirovskii , Phys. Rev. B57 (1997), 5792.

[10] S. K. Nemirovskii , Phys. Rev. Lett. 96 (2006), 015301.

[11] K. W. Schwarz, Phys. Rev. B38 (1988), 2398.

[12] W. F. Vinen, Phys. Rev. B71 (2005), 024513.
[13] W. F. Vinen and J. J. Niemela, J. Low Temp. Phys. 516 (2002), 167.
[14] G. E. Volovik, JETP Lett. 78 (2003), 1021.