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SINGULARITIES IN 3D LAMINAR BOUNDARY LAYER AND FLOW STRUCTURE NEAR A SINK PLANE ON CONICAL SURFACES

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Abstract. The flow over conical bodies is a simplest and well-studied problem of 3D laminar boundary layer theory. However found here singularities, which have direct relation to such problems as boundary-layer solution unique existence, and 3D separation does not still explained. In this work, the singularities arising in the outer part of the boundary layer in the sink plane on pointed conical bodies are studied. This singularity generates the asymptotic flow structure that includes: the boundary region, in which reduced Navier-Stokes equations describe the flow; the region, in which the viscous-inviscid interaction is main effect and two-layer asymptotic model is appropriate to flow description. Analytical solutions for these regions are obtained and analyzed.

1. SINGULARITIES IN 3D BOUNDARY LAYER ON CONICAL SURFACES

The 3D laminar boundary layer on a conical surface for linear viscosity dependence on temperature and Prandtl number $\Pr = 1$ in the orthogonal coordinate system $xy\varphi$ (Figure 1) is described by the equations and boundary conditions[1]

$$u_{yy} = Awu_\varphi - vu_y + A_1 w(u - w),\ h_{yy} = Awh_\varphi - vh_y - M_0\left(u_y^2 + \tfrac{3}{2}A_1 w_y^2\right),\ \rho h = 1,$$

$$w_{yy} = Aww_\varphi - vw_y + w\left(\tfrac{2}{3}u + Kw\right) - h\left(\tfrac{2}{3} + K\right),\ v = f + \left[K - \tfrac{1}{2}A\left(\ln\left(\rho_e\,\mu_e/u_e\right)\right)'\right]g + Ag_\varphi,$$

$$f_y = u,\ g_y = w;\ y = 0: u = v = w = 0,\ h = h_w\ (h_y = 0);\ y = \infty: u = w = h = 1, \quad (1.1)$$

$$A(\varphi) = \frac{2w_e}{3Ru_e},\ A_1(\varphi) = \frac{2}{3}\left(\frac{w_e}{u_e}\right)^2,\ K(\varphi) = \frac{2w_e'}{3Ru_e},\ M_0(\varphi) = \frac{u_\infty^2 u_e^2}{h_\infty h_e},$$

$$y = \sqrt{\frac{3\,\mathrm{Re}}{2x}}\int_0^{y^*}\rho\,dy^*,\ \mathrm{Re}(\varphi) = \frac{\rho_e u_e \mathrm{Re}_\infty}{\mu_e},\ \mathrm{Re}_\infty = \frac{\rho_\infty u_\infty l}{\mu_\infty}.$$

Here x is the distance along generator referenced to the body length l; φ is transversal coordinate; $f(y,\varphi)$, $g(y,\varphi)$ and $v(y,\varphi)$ are flow functions and transformed normal velocity; y^* is referenced to l normal to the body; $R(\varphi)$ is metric coefficient; asterisk and indexes x, φ and y denote the differentiation with respect to argument and to corresponding variables. The density ρ, enthalpy h, viscosity μ, longitudinal (u) and transversal velocities (w), are referenced to them values at the outer boundary. The functions on the boundary-layer edge indexed by "e" are normalized to them freestream values indexed by "∞"; they are functions of φ only. The transversal velocity $w_e = 0$ in the source plane $\varphi = 0$ ($K(0) > 0$), and in the sink plane $\varphi = \varphi_1$ ($K(\varphi_1) = -k < 0$), in which two boundary layer parts came from different sides of the source plane are collided.

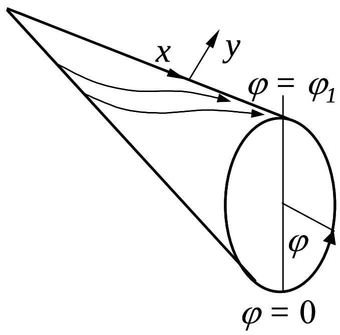


Figure 1 Flow scheme and coordinate system.

For slender bodies, Eqs. (1.1) are simplified since $A = O(1)$, $A_1 << 1$, $u_e = \rho_e = \mu_e = 1$. In this case the enthalpy equation (1.1) allows the Crocco integral

$$h = h_w + h_r u - \tfrac{1}{2} M_0 u^2,\ h_r = 1 - h_w + \tfrac{1}{2} M_0 . \tag{1.2}$$

We consider the asymptotic of Eqs. (1.1)-(1.2) at $y >> 1$. In this region, functions are represented as

$$\begin{gathered} u = 1 + U(\eta,\varphi),\ w = 1 + W(\eta,\varphi),\ H = -\left(\tfrac{1}{2} M_0 + h_w - 1\right) U,\ v = (1+K) y, \\ U_{\eta\eta} + \eta U_\eta - aAU_\varphi = 0,\ W_{\eta\eta} + \eta W_\eta - \tfrac{2}{3} a\left[\tfrac{3}{2} AW_\varphi + (1+3K) W + pU\right] = 0, \\ \eta = y\big/\sqrt{a(\varphi)},\ N(\varphi) = 3M(\varphi) = K^{-1},\ p(\varphi) = 1 + \left(1 + \tfrac{3}{2} K\right)\left(\tfrac{1}{2} M_0 + h_w - 1\right), \\ U(\eta,\varphi) = C_1 erfc\left(\eta/\sqrt{2}\right),\ a' + 2\left[(N+1)\left(\ln w_e \sqrt{u_e/\rho_e\mu_e}\right)'\right] a = 2N(\ln w_e)', \\ W(\eta,\varphi) = -b(\varphi) U,\ b' + 2(1+M)(\ln w_e)' b = 2pM(\ln w_e)'. \end{gathered} \tag{1.3}$$

Here $C_1(k)$ is constant. Solutions for the functions $a(\varphi)$ and $b(\varphi)$ have the form

$$m \neq 1 : b = \frac{Mp}{M+1} - Ew_e^{-2(M+1)} \int_0^\varphi \frac{p'M(M+1) + pM'}{E(M+1)^2} w_e^{2(M+1)} d\varphi ,$$

$$E = \exp\left(2\int_0^\varphi M'(t) \ln w_e(t)\, dt\right),$$

$$m=1:b=2Mp\ln w_e-2\int_0^{\varphi}\frac{(pM)'\,w_e+2(M+1)Mpw_e'}{E}w_e^{2M+1}\ln w_e d\varphi\,, \qquad (1.4)$$

$$n\neq 1:a=\frac{N}{N+1}-E_1w_e^{-2(N+1)}\int_0^{\varphi}\frac{N'}{E_1(N+1)^2}w_e^{2(N+1)}d\varphi\,,\ E_1=\exp\left(2\int_0^{\varphi}N'(t)\ln w_e(t)\,dt\right),$$

$$n=1:a=2N\ln w_e-2E_1w_e^{-2(N+1)}\int_0^{\varphi}\frac{N'w_e+2(N+1)Nw_e'}{E_1(N+1)^2}w_e^{2N+1}\ln w_e d\varphi\,.$$

These solutions satisfy to initial conditions at $\varphi=0$ for regular at $K(0)\to 0$ solution branch[2], the expressions for $n=m=1$ are true at $\varphi>0$ only. At $\zeta=\varphi_1-\varphi<<1$

$$w_e=\tfrac{3}{2}kR\zeta+O\left(\zeta^2\right),\ k=-K\left(\varphi_1\right),\ R=R\left(\varphi_1\right),\ p_1=p(\varphi_1)\,,$$

$$m\neq 1:b=\frac{mp_1}{m-1}-b_m\zeta^{2(m-1)};\ m=1:b=-2p_1\ln\zeta+b_1;\ m=-M\left(\varphi_1\right), \qquad (1.5)$$

$$n\neq 1:a=\frac{n}{n-1}+a_n\zeta^{2(n-1)};\ n=1:a=-2\ln\zeta+a_1;\ n=-N\left(\varphi_1\right).$$

The coefficients a_n and b_m are determined by integrals (1.4) with $\varphi=\varphi_1$. The formulas (1.5) are true for non-slender bodies also since $A_1=O(\zeta^2)$.

There are two singularity types in the boundary layer. For $k<1$ the function $U\left(\eta,\zeta\right)$ exists at $\zeta=0$ but is irregular. At $k\geq 1$ the function $U\left(\eta,\zeta\right)$ is singular at $\zeta=0$, the boundary layer infinitely growths as $\sqrt{a\left(\zeta\right)}$ at $\zeta\to 0$; at $k=1$ the singularity is of logarithmic type. The function $W\left(\eta,\zeta\right)$ is irregular at $\zeta\to 0$: it has finite limit at $k<1/3$, and is singular at $k\geq 1/3$; for $k=1/3$ the singularity is of logarithmic type. The irregularity type of both functions is studied in details for the slender round cone[2].

2. THE BOUNDARY REGION FLOW

The singularity of the boundary-layer equations leads to vortex boundary region formation of transversal dimension of the order of boundary layer thickness. In this region, the transverse diffusion is of the first order effect, and to describe it we introduce the variables

$$z=\sqrt{kx}R\zeta/\varepsilon\,,\ u=u\left(y,z\right),h=h\left(y,z\right),\ w=w\left(y,z\right),\ \varepsilon=\left[\tfrac{3}{2}\mathrm{Re}\left(\varphi_1\right)\right]^{-\frac{1}{2}}.$$

The principal approximation of the Navier-Stokes equations at $\varphi\to\varphi_1$, $\varepsilon\to 0$, and $y>>1$ has the form

$$\begin{gathered}U_{yy}+kU_{zz}+\left(1-k\right)yU_y+kzU_z=0,\\ W_{yy}+\left(1-k\right)yW_y+kW_{zz}+\left(\frac{2}{z}+kz\right)W_z+2k\left(m-1\right)W+\tfrac{2}{3}p_1U=0.\end{gathered} \qquad (2.1)$$

For $k<1$ these equations has the solution

$$U(y,z) = C_1 erfc\left(y\sqrt{(1-k)/2}\right) erf\left(z/\sqrt{2}\right),\ W = -B(z) C_1 erfc\left(y\sqrt{(1-k)/2}\right),$$

$$B_{zz} + \left(\frac{2}{z} + z\right) B_z - 2(m-1) B = -2mp_1 F(z),\ F(z) = erf\left(z/\sqrt{2}\right), \quad (2.2)$$

$$B = mp_1 B_0(z) + B_m \Phi\left(1-m, \tfrac{3}{2}, -\tfrac{1}{2}z^2\right),\ B_m = b_m \left(R\sqrt{kx}/\varepsilon\right)^{2(1-m)}.$$

,

Here $\Phi(a,b,x)$ is Kummer function, and $B_0(z)$ is particular solution of inhomogeneous equation. The coefficient B_m is determined from matching condition of (2.2) with (1.5). In the particular case $m=1$, (2.2) is integrated in the explicit form

$$B = B_1 - 2p_1 F_1(z), \quad B_1 = b_1 + p_1\left(2\ln\left(R\sqrt{kx}/\varepsilon\right) + C + \ln 2 - 1\right),$$

$$F_1(z) = F(z)\left(\ln z - \tfrac{1}{2}\right) - \sqrt{\tfrac{2}{\pi}}\int_0^z e^{-1}(t)\ln t\,dt - \int_0^z e^{-1}(x)x^{-2}\int_0^x e(t)F(t)dt, \quad e(z) = \exp\left(z^2/2\right).$$

where C is Euler constant. The function $F_1(z)$ and its boundary-layer asymptote at $z \gg 1$, $\ln z - 0.54$, are denoted by 1 and 2 in Fig. 2. Another explicit solution corresponds to $m = 1/2$

$$B(z) = B_{1/2} F(z)/z - 2p_1\left\{F(z) + 2\sqrt{\tfrac{2}{\pi}}\left[e^{-1}(z) - 1\right]/z\right\}.$$

The function $F(z)/z$ and its asymptote $1/z$ are denoted in Fig. 2 by 3 and 4.

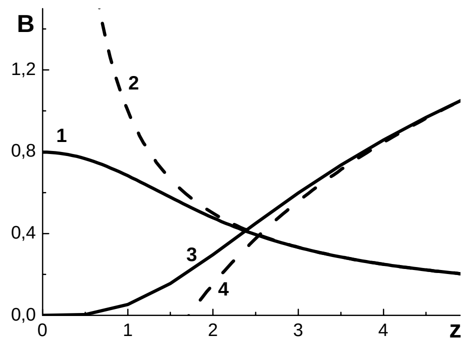


Figure 2. Solutions for the boundary region: lines 1 and 3 corresponds to $m=1$ and $m=1/2$; lines 2 and 4 are them asymptotes in the boundary layer.

Therefore, at $k \ge 1/3$ near the sink plane the vortex region is formed, in which the reduced Navier-Stokes equations describe the flow. Obtained solutions are regular and matched with solutions of the boundary layer equations.

3. THE REGION OF VISCOUS-INVISCID INTERACTION

The singularity obtained leads also to the boundary layer growth at $\zeta \to 0$ and appearance of the viscous-inviscid interaction. From (1.3) and (1.5) the estimations for the boundary layer thickness $\Delta(\zeta)$ can be obtained in the form

$$k=1/3:\Delta\sim\sqrt{\ln\ln\left(1/\zeta^2\right)};\ 1/3<k<1:\Delta\sim\sqrt{(1-m)\ln\left(1/\zeta^2\right)/(1-k)},$$
$$k>1:\Delta\sim\zeta^{n-1}\sqrt{\ln\left(1/\zeta^2\right)};\ k=1:\Delta\sim\ln\left(1/\zeta^2\right). \tag{3.1}$$

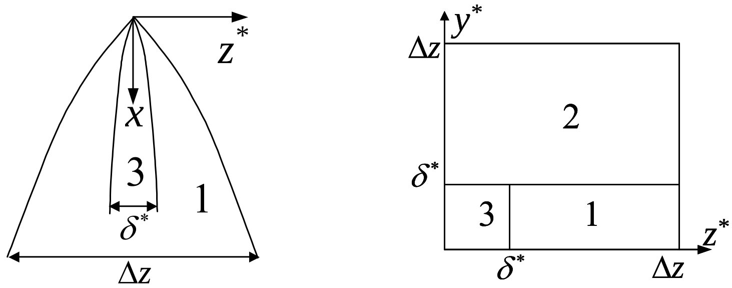
In order to the viscous-inviscid interaction effect would be of principal order the transversal velocity w_e from (1.1) would be of the order of the velocity w_{ei} induced by the boundary-layer growth. This condition allows estimating the transverse dimension of the interaction region $\Delta\varphi$ as

$$w_{ei}\sim\frac{2\varepsilon u_e}{R\sqrt{x}}\zeta\frac{\partial\Delta}{\partial\zeta}\sim w_e=\tfrac{3}{2}kRu_e\Delta\varphi,\ \Delta\varphi\sim\zeta\sim\sqrt{m\varepsilon}\,x^{-\frac{1}{4}},\ w_e\sim w_{ei}\sim kRu_e\sqrt{m\varepsilon}\,x^{-\frac{1}{4}}. \tag{3.2}$$

In Region 1 in Fig. 3 ($y\sim1$, $\zeta\sim\Delta\varphi$), the flow is described by the equations

$$s=R\zeta/\sqrt{\varepsilon},\ u_e=u_e(\varphi_1)+O(\varepsilon),\ h_e=h_e(\varphi_1)+O(\varepsilon),\ w_e=\tfrac{3}{2}u_e\sqrt{\varepsilon}W_e(x,s),\ A=W_e,$$
$$K=W_{es},\ A_1=O(\varepsilon),\ R=R(\varphi_1)+O(\varepsilon),\ v=f+Kg+Ag_s+\tfrac{2}{3}xf_x,\ u=u(x,y,s), \tag{3.3}$$
$$h=h(x,y,s),\ u_{yy}=W_ewu_s-vu_y+\tfrac{2}{3}xuu_x,\ h_{yy}=W_ewh_s-vh_y-M_0u_y^2+\tfrac{2}{3}xuh_x$$
$$w=w(x,y,s),\ w_{yy}=W_eww_s-vw_y+w\left(\tfrac{2}{3}u+W_{es}w\right)-h\left(\tfrac{2}{3}+W_{es}\right)+\tfrac{2}{3}xuw_x.$$

The boundary conditions for Eqs. (3.3) have the form (1.1), and the enthalpy equations is integrated in the form (1.2). Solutions of Eqs. (3.3) will be matched with (1.5) at $s\to\infty$. Initial conditions are needed also for Eqs. (3.3) an some $x=x_0$.



a) top view; b) back view

Figure 3. Flow structure near the sink plane

In Region 2 in Fig. 3 ($y^*\sim\zeta\sim\sqrt{\varepsilon}$), the flow is inviscid and at moderate supersonic Mach numbers is nonvortical. On the body surface we obtain

$$W_e(x,s)=-ks[1+r],\ q(x,s)=-\frac{4m}{\pi}\frac{\partial}{\partial x}\int_0^\infty\frac{\delta_t(x,t)tdt}{s^2-t^2},\ r(x,s)=\frac{4m}{\pi}\frac{\partial}{\partial x}\int_0^\infty\frac{\delta(x,t)dt}{s^2-t^2}.$$

In the outer boundary layer part at $y>>1$, the solution of (3.3) has the form

$$U_{tt}+tU_t+k(1+r)dsU_s-\tfrac{2}{3}dxU_x=0,\quad U=C_1erfc\left(t/\sqrt{2}\right),\ r(x,s)=\frac{4m}{\pi}\frac{\partial}{\partial x}\int_0^\infty\frac{\delta(x,t)dt}{s^2-t^2},$$
$$(1+r)sd_s-2mxd_x-2(n-1+q)d=-2n,\ q(x,s)=-\frac{4m}{\pi}\frac{\partial}{\partial x}\int_0^\infty\frac{\delta_t(x,t)tdt}{s^2-t^2},$$
$$(1+r)sc_s-2mxc_x-2(m-1+q)c=-2m(p_1-qp_0),\quad p_0=\tfrac{3}{2}\left(\tfrac{1}{2}M_0+h_w-1\right),$$

$$t = y\big/\sqrt{d(x,s)},\ u = 1 + U(x,t,s),\ w = 1 - c(x,s)U, \quad v = y\left[1 - k(1+r)\right].$$

Along characteristics $\xi(x,s) = const$ ($(1+r)s\xi_s - 2mx\xi_x = 0$), which are streamlines of the inviscid flow, the equations are reduced to the equations similar to (1.4)

$$(1+r)sd_s - 2(n-1+q)d = -2n, \quad (1+r)sc_s - 2(m-1+q)c = -2m(p_1 - qp_0),$$

$$c = CE_2 s^{L-1} + Q(\xi,s) + s^{L-1}E_2\int_s^\infty Q_t E_2^{-1} t^{1-L}dt, \quad E_2(\xi,s) = \exp\left(-\int_s^\infty L_t \ln t dt\right),$$

$$d = DE_3 s^{I-1} + \frac{n}{n-1+q} - ns^{I-1}E_3\int_s^\infty \frac{q_t E_3^{-1} t^{1-I} dt}{(n-1+q)^2}, \quad E_3(\xi,s) = \exp\left(-\int_s^\infty I_t \ln t dt\right), \tag{3.4}$$

$$I(\xi,s) = \frac{n-1+q}{1+r}, \quad L(\xi,s) = \frac{m-1+q}{1+r}, \quad Q = \frac{m(p_1 - p_0 q)}{m-1+q}.$$

At $s \to \infty$ the interaction is became weaker, $r \to 0$ и $q \to 0$, and the integrals in Eqs. (3.5) tend to zero. From matching conditions we find: $D = a_n\varepsilon^{n-1}$, $C = b_m\varepsilon^{m-1}$.

4. CONCLUSIONS

Explicit solutions of equations for the outer boundary layer part on slender conical bodies are obtained. The singularity of the solutions leads to formation of multi-layer flow structure near the sink plane. In the boundary region 3 the flow is described by reduced Navier-Stokes equations with the constant pressure gradient. The obtained solutions of these equations are regular and matched with solutions of boundary layer equations. In Region 2, the flow is inviscid and determined by interaction with boundary layer. The 3D interacting boundary layer lies at the bottom of Region 2 (Region 1). In whole singular region including Regions 1, 2, and 3, the flow is described by parabolized Navier-Stokes equations, which are composite equations from the point of view of asymptotic theory. Results of item 3 show that singularities of regarded type can occur in solutions of the 3D boundary layer equations on arbitrary bodies.

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5. REFERENCES

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