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WEAKLY NONLINEAR PULSATION MODELS IN LAMINAR AND TURBULENT BOUNDARY LAYERS

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Abstract. Time dependent laminar or statistically stationary turbulent flow in incompressible fluid on the flat plate at zero angle of attack is considered without a longitudinal pressure gradient in the frame of Navier-Stokes equations. Developed turbulent motion on a plate has good defined average flow. So we can apply the method that we have used in laminar boundary layer. In the turbulent boundary layer the explanation of streak structure and value of bursting time period was given on the base of multiple 3-wave resonance. Besides we obtain scaling law for the boundary thickness ratio.

1. INTRODUCTION

For the study of turbulent dynamics of fluid or gas two types of models are used. They are Boussinesq-Prandtl and Smagorinsky subgrid (LES) models. The former models are closed equations relatively some functions of mean motion and can be written for steady case. The latter equations are substantially time dependent, therefore moment coefficient statistics is collected in the process of long time numeric simulation data analyses. At present time the second type models are improving greatly. It is connected with the fact that we have now the experimental data of deep physical sense. As a result the nonlinear dependence of stress tensor on strain rate tensor was modeled both theoretically (RNG method) and phenomenologicaly.

Both model types have some disadvantages. The first type models require experimental boundary conditions on the wall (the law of the wall) and additional experimental data for simulation of the so called turbulent viscosity. Large eddy models got by RNG methods have differential order exceeding physical boundary condition number (stress tensor includes space average velocity derivatives of the third order). As a result we must use additional nonphysical boundary conditions.

According to experimental researches[1,2] there exist the regions near by the wall which depends on longitudinal vorticity. The vorticity intensity is approximately 10 times less than average transverse vorticity of the flow in turbulent boundary layer. Small parameter follows from this fact. (By the way, this small parameter was used multiply for deducing logarithmic dependence of average longitudinal velocity on transverse coordinate in buffer sublayer[3]).

All the above mentioned gives us an opportunity to develop physical approach to description of these structures on the base of multiple 3-wave resonance[4], making use of small parameter in original equations. The theory in this case is weakly nonlinear. It contradicts to some extent the current representations in this field. However, as mentioned in paper[5], there exists an analogy between the properties of laminar and turbulent boundary layer. Analogies of paper[5] can be continued because there exists quite definite "average" field both in laminar and developed turbulent boundary layer. And the disturbances develop on the background of this average field. Small parameter finishes the analogy. In present paper weakly nonlinear theory of turbulent boundary layer with coherent structures extraction is developed.
Further experimental weakly nonlinear properties of laminar and turbulent boundary layers are considered, a correspondence with results of weakly nonlinear theory is established, and similar weakly nonlinear theory of turbulent boundary layer with extraction of coherent structures is developed[6].

2. DEFINING EXPERIMENTAL DATA COLLECTION

In laminar boundary layer the following experimental data is known: continuum low frequency spectrum[7, 8]; the wave packet wake[9]; wave packet dynamics[10]; formation of strip–like structures, the development of which bring to high frequency wave packet appearance[11]. These phenomena are perfectly explained by weakly non-linear theory, which is shown in papers[12,13,14]. The solution of 3-wave resonance equations for the Blasius boundary layer is shown in Fig.1 (б). The weakly non-linear theory applicability is bounded[13] (pulsation and viscosity stress have the same order) by the relation $\max_{\mathbf{k}} \operatorname{Im} \omega(\mathbf{k}, R) \le q/\sqrt{R},\ q = O(1)$, here $\omega(\mathbf{k}, R)$ is eigenvalue of the lowest Orr-Sommerfeld mode, $\mathbf{k}$ is wave number.

In turbulent boundary layer it is necessary, from our point of view, to select the following fundamental experimental facts: the presence of mean profile and structure of longitudinal velocity component, containing logarithmic sublayer; the presence of time dependent coherent structures, the similarity law for burst frequency ($TU_\infty/\delta = const$, T - mean burst period, δ - turbulent boundary layer thickness, U_∞ - free stream velocity)[1]; longitudinal vortex magnitude is equal to ~1/10 of transversal vortex magnitude[1]; known small parameter[3] $\varepsilon^2 = \delta^{**}/L \sim \tau_w/2 \approx 0.01$, L - typical longitudinal scale, δ^{**} - momentum thickness; τ_w - non-dimensional tangent wall stress[3]; well-developed turbulent condition $R >> 1, \varepsilon^2 R >> 1$, R - Reynolds number by momentum loss thickness; triple decompositon of flow field[15]; the presence of energy backscattering along spectrum[16];

3. THEORETICAL RESULTS

The apparatus of weakly nonlinear theory was created during last decades. This apparatus contains solution of Orr-Sommerfeld (O–S) and Squire (S) equation spectral problem within semi-infinite interval, continuum and discrete spectrum eigenfunctions, orthogonality and completeness of these eigenfunctions[17], peculiarity of O–S eigenfunction at small wave numbers[18], dispersion relation of 3–D Tollmin-Schlichting (T–S) and Squire (S) waves and their peculiarities[19,20,21], wave packets, the wave packet envelope equation, matrix elements, matrix element peculiarities at small wave numbers[20,21,22,23]; multiple 3-wave resonance[24,25]; correlated function representation, initial–value problem for correlation functions, multiscale method for correlation function equations[26].

4. ORIGIN EQUATIONS

The velocity field is decomposed into two parts: the time mean part and the pulsation part. Equations for coherent part can be separated from equations for stochastic pulsations due to presence of 3–wave resonance of discrete spectrum T-S waves, i.e. as a result we have triple decomposition[15] of the velocity field of a turbulent flow. For the incoherent part we have in general case the system of kinetic equations which can be deduced to one equation at definite assumptions[6].

Let d be typical transversal scale of the flow, L - longitudinal. The value of $d \sim \delta^{**}$, where δ^{**} is momentum thickness. U и V are mean longitudinal and transversal velocity components, and u, v, w, p are pulsation velocity components and pressure, U_∞ is free stream velocity, ν - kinematic viscosity. We shall consider dimensionless values: $\overline{U} = U/U_\infty$, $\varepsilon^2 \overline{V} = V/U_\infty$, $\varepsilon \overline{u}_i = u_i/U_\infty$, $\varepsilon \overline{p} = p/\rho U_\infty^2$, $\overline{x}_i = x_i/d$, $\overline{t} = tU_\infty/d$, $X = x/L$, $T = tU_\infty/L$, $\{u_i\} = (u, v, w)$, $\{x_i\} = (x, y, z)$, $i = 1,2,3$, $R = U_\infty d/\nu$. Then the boundary layer approximation equations for $\overline{U}$ and $\overline{V}$ can be written as

$$\frac{\partial \overline{U}}{\partial T} + \overline{U}\frac{\partial \overline{U}}{\partial X} + \overline{V}\frac{\partial \overline{U}}{\partial \overline{y}} = \frac{\partial}{\partial \overline{y}}\left(-\langle \overline{u}\overline{v}\rangle + \frac{1}{\varepsilon^2 R}\frac{\partial \overline{U}}{\partial \overline{y}}\right), \quad \frac{\partial \overline{U}}{\partial X} + \frac{\partial \overline{V}}{\partial \overline{y}} = 0,$$

and for $\overline{u}_i, \overline{p}, i = 1,2,3$,

$$\frac{\partial \ \overline{u}_i}{\partial \ \overline{t}} + \overline{U}\frac{\partial \ \overline{u}_i}{\partial \ \overline{x}_1} + f = -\frac{\partial \ \overline{p}}{\partial \ \overline{x}_i} + \frac{1}{R}\nabla^2 \overline{u}_i + \varepsilon \ \ T_i + o\left(\varepsilon^2\right), \quad \frac{\partial \ \overline{u}_i}{\partial \ \overline{x}_i} = 0 \qquad (1)$$

We use the following notations in equation (1): $\{f_i\} = (\overline{v}\left(\partial \overline{U}/\partial \overline{x}_2\right), 0, 0)$, $T_i = \partial\left(\langle \overline{u}_i \overline{u}_j\rangle - \overline{u}_i \overline{u}_j\right)/\partial \overline{x}_j$. Boundary conditions of system (1) are $\overline{u}_i = 0$, $i = 1, 2, 3$, at $\overline{y} = 0, \infty$ and $\overline{p} = 0$ at $\overline{y} = \infty$. Reynolds number R satisfies conditions $R >> 1$ and $\varepsilon^2 R >> 1$.

Pulsation velocity equations are equivalent to two equations: O–S equation for vertical velocity component and S – equation for vertical vorticity component. Due to completeness of their eigenfunctions the solution can be represented as series of the eigenfunctions.

5. EQUATIONS FOR AMPLITUDES IN ONE-MODE APPROXIMATION

Let us develop the vertical velocity and the vertical vorticity as series in eigenfunctions of O–S and S – equations respectively $\overline{v}_{\mathbf{k}}(y) = A_{\mathbf{k}}^{(0)}\varphi_{\mathbf{k}}^{(0)}(y) + \dots$, $\overline{\eta}_{\mathbf{k}}(y) = \hat{\eta}_{\mathbf{k}}^{ind} + \dots$. Here $\overline{v}_k$, $\overline{\eta}_k$ are Fourier images of vertical velocity and the vertical vorticity, sums are taken over discrete spectrum modes (eigenfunctions of O–S and S – equations), and the integral – over the continuum modes. In one–mode approximation we have: $\overline{v}_{\mathbf{k}}(y) = A_{\mathbf{k}}\varphi_{\mathbf{k}}^{(0)}(y) + \dots$, $\overline{\eta}_{\mathbf{k}}(y) = -\beta \overline{v}_{\mathbf{k}} / \left(\alpha\left(U(y) - c_{\mathbf{k}}\right)\right)\partial U(y)/\partial y + O(1/R) + \dots$,

$\bar{u}_{\mathbf{k}}(y)= \quad i\left(\alpha A_{\mathbf{k}} d\varphi_{\mathbf{k}}^{(0)}(y)/dy-\beta\hat{\eta}_{\mathbf{k}}(y)\right)/k^2+\ldots, \quad \bar{w}_{\mathbf{k}}(y)=i\left(\beta A_{\mathbf{k}} d\varphi_{\mathbf{k}}^{(0)}(y)/dy+\alpha\hat{\eta}_{\mathbf{k}}(y)\right)/k^2+\ldots$. As a result the solution for coherent and stochastic parts can be represent as

$$v_{\mathbf{k}\omega}=A_{\mathbf{k}}\varphi_{\mathbf{k}}^{(0)}\delta\left(\omega-\omega_{\mathbf{k}}^{(0)}\right), \quad \mathbf{k}_1^0=\mathbf{k}_2^0+\mathbf{k}_3^0, \; A_{-\mathbf{k}}=A_{\mathbf{k}}^*, \; A_{\mathbf{k}}=A_{\mathbf{k}}^c+A'_{\mathbf{k}}, \quad A_{\mathbf{k}}^c=b\delta(\mathbf{k})+a_1\delta\left(\mathbf{k}-\mathbf{k}_1^0\right)+$$
$$+\sum\left\{a_2\delta\left(\mathbf{k}-\mathbf{k}_2^0\right)+a_3\delta\left(\mathbf{k}-\mathbf{k}_3^0\right)\right\}+a_1^*\delta\left(\mathbf{k}+\mathbf{k}_1^0\right)+\sum\left\{a_2^*\delta\left(\mathbf{k}+\mathbf{k}_2^0\right)+a_3^*\delta\left(\mathbf{k}+\mathbf{k}_3^0\right)\right\}, \quad \left\langle A'_{\mathbf{k}}A'_{\mathbf{k}_1}\right\rangle=$$
$$=\Gamma_2(\mathbf{k})\delta(\mathbf{k}+\mathbf{k}_1), \quad \left\langle A'_{\mathbf{k}}A'_{\mathbf{k}_1}A'_{\mathbf{k}_2}\right\rangle=\varepsilon\Gamma_3(\mathbf{k},\mathbf{k}_1)\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2), \quad c_{\mathbf{k}}=\omega_{\mathbf{k}}^{(0)}/\alpha.$$

The sum in the equation for $A_{\mathbf{k}}^C$ is taken along all resonance triplets. This representation gives us nonlinear equation for the coherent part $A_{\mathbf{k}}^C$ and equations for $\Gamma_2(\mathbf{k})$ and $\Gamma_3(\mathbf{k},\mathbf{k}_1)$. The equation of the coherent part contains the integral from the correlated function $\Gamma_2(\mathbf{k})$ over all wave numbers. This provide for the backscatter process. Equations for $A_{\mathbf{k}}^C$, $\Gamma_2(\mathbf{k})$ and $\Gamma_3(\mathbf{k},\mathbf{k}_1)$ contain matrix elements[6]. Besides it is supposed the relation $\delta^{**}/L\sim\max_{\mathbf{k}}\left|\mathrm{Im}[\omega_{\mathbf{k}}]\right|\sim\varepsilon^2$ is carried out. On the assumption that the wave vectors of the coherent part are near by the coordinate origin of the wave number space and that, for stochastic component, the difference of this wave vector from zero is inessential, we can consider these equations as exact. It brings us the kinetic type equation for stochastic component. Since the system of equations contains the small parameter let us decompose looked for values and time derivative in series by ε using multyscale method: after substitution of series in these equations we have that in considered orders amplitudes of the coherent structure don't depend on t_0, and function $\Gamma_2[\mathbf{k}]$ don't depends on t_0, t_1. Make a note here, that in the equation for incoherent part we must take into consideration the average part of amplitudes of time dependent

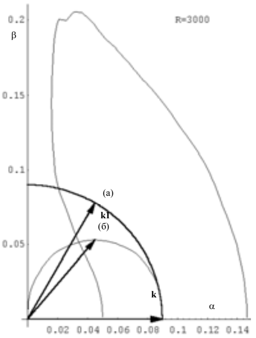

Fig. 1.

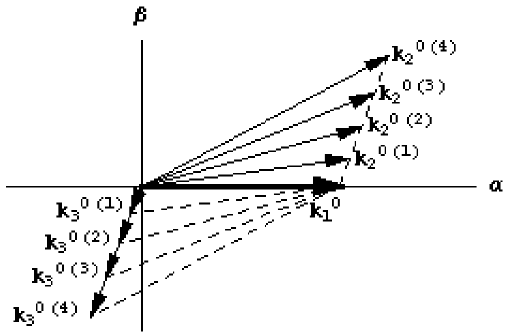

Fig. 2.

system of equations for coherent structure at scale $t_1=\varepsilon\cdot t_0$. As a result parts of the stress tensor $\sigma_{xy}=-\langle uv\rangle=\sigma'_{xy}+\sigma_{xy}^C$ are the following:

$$\sigma'_{xy}=\frac{2}{(2\pi)^6}\int_0^\infty d\alpha\int_{-\infty}^\infty d\beta\Gamma_2^{(0)}(\mathbf{k})\left[\frac{\alpha}{k^2}\mathrm{Im}\left(\varphi_{\mathbf{k}}^{*(0)}\partial\varphi_{\mathbf{k}}^{(0)}/\partial y\right)-\frac{\beta^2\left|\varphi_{\mathbf{k}}^{(0)}\right|^2\mathrm{Im}(c_{\mathbf{k}})}{k^2\alpha\left|U(y)-c_{\mathbf{k}}\right|}\frac{\partial U(y)}{\partial y}\right],$$

$$\sigma_{xy}^C=\frac{2}{(2\pi)^6}\sum_{s=1}^3\left(\frac{|a_s|^2\alpha_s^0\,\mathrm{Im}[\varphi_0(\mathbf{k}_s^0)\partial\varphi_0^*(\mathbf{k}_s^0)/\partial y]}{\left|\mathbf{k}_s^0\right|^2}-\frac{|a_s|^2\left(\beta_s^0\right)^2\left|\varphi_0(\mathbf{k}_s^0)\right|^2\mathrm{Im}[c(\mathbf{k}_s^0)]}{\left|\mathbf{k}_s^0\right|^2\alpha_s^0\left|U(y)-c(\mathbf{k}_s^0)\right|^2}\frac{\partial U(y)}{\partial y}\right)+\ldots$$

where $\mathbf{k}_s^0 = (\alpha_s, \beta_s), s = 1,2,3,$ satisfy to 3–wave resonance condition. Due to $\mathbf{k}_1^0 = (\alpha_1, 0)$ the member, proportional to $\partial U / \partial y$, at $s = 1$, is absent. In the case of multiple 3-wave resonance σ_{xy}^C will contain the sum of members over all resonance triplets (Fig. 2).

6. CONCLUSIONS

1) Tensor σ_{xy} depends on time on $T_1 = \delta^{**} / (\varepsilon U_\infty) = \delta / U_\infty$ scale; 2) connection between stress tensor σ_{xy} and average velocity characteristics is **functional**; 3) σ_{xy} depends only on $U(y)$, $U'(y)$ and $U''(y)$; 4) turbulent stress tensor consisting of superposition of coherent and random part $\sigma_{xy} = \sigma_{xy}^{(c)} + \sigma'_{xy}$, can be represented as $a(y) + b(y)\partial U / \partial y + O(1/R)$, and tends to zero at $y \to 0$; 5) the part σ_{xy} proportional to $\partial U / \partial y$ is created by the correlation between vertical velocity and vertical vorticity, i.e. T–S and S–waves; 6) the main contribution in $a(y)$ is defined by T- S waves with $\mathbf{k}$ along the flow, the main contribution in $b(y)$ is defined by T- S waves with $\mathbf{k}$ transverse to flow (the curve of the 3- wave resonance for turbulent velocity profile is given in[23]); 7) there exists subrange[19] were the logarithmic dependence of velocity profile on y takes place.

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