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ON THE STRUCTURE OF SHOCK WAVES IN A TWO-PHASE ISOTHERMAL MODEL

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Abstract. A traveling wave analysis of a two phase isothermal Euler model is performed in this work. This analysis allows to exhibit the inner structure of shock waves in two-phase flows. In the model under investigation, the dissipative regularizing term is not of viscous type but instead comes from relaxation phenomena toward equilibrium between the phases. This gives an unusual structure to the diffusion tensor where dissipative terms appear only in the mass conservation equations. We show that this implies that the mass fractions are not constant inside the shock although the Rankine-Hugoniot relations give a zero jump of the mass fraction through the discontinuities. We also show that there exists a critical speed for the traveling waves above which no $\mathscr{C}^1$ solutions exist. Neverthless for this case, it is possible to construct traveling solutions involving single phase shocks.

1. INTRODUCTION

The correct definition of shock solutions in two-phase models is an open question. Actually, many two-phase models are in non-conservative form and in addition to the classical relations expressing the conservation of mass, momentum and energy contain additional equations in non conservative form. In principle, traveling waves analysis[1,5] provide a satisfactory way to describe the inner structure of a shock and consequently should allow a rigorous definition of shock solutions for these systems. However, the practical realization of a travelling wave analysis requires to identify the precise shape of the dissipative tensor. Actually, the regularizing effect of this tensor precisely dictate the amplitude of the jump relations connecting the two states of the discontinuity. In the framework of two-phase system, this implies that the dissipative tensor cannot be arbitrary and must in some sense encode the right physic of the inner structure of a two-phase shock.
Many works on the regularisation of non-conservative hyperbolic systems consider viscous effects as the leading mechanism allowing the definition of shock waves.

However, in two phase systems, we would like to emphasize that other effects than viscous regularization can exist. We will present here another kind of these possible regularizing effects based on the existence of relaxation phenomena in two-phase systems that drive the two phases toward mechanical and thermodynamical equilibrium. The existence of these relaxation mechanisms gives a particular and unusual structure to the dissipative tensor. We will see that a traveling wave analysis of the dissipative system shows some unusual consequences on the structure of shock waves in two-phase flows.

2. THE TWO-PHASE EULER ISOTHERMAL MODEL

The model we consider in this work has been introduced in ref[2] as a model for isothermal dispersed bubbly flows. We refer to this work for its derivation. Numerical simulations performed with this model has shown that despite its simplicity, it is able to reproduce two-phase computations usually performed with more complex models. In one dimension, this system can be written

$$\begin{aligned}
&\frac{\partial}{\partial t}(\rho) + \frac{\partial}{\partial x}(\rho u) &= 0 \quad (1.1)\\
&\frac{\partial}{\partial t}(\rho Y) + \frac{\partial}{\partial x}(\rho Y u) - \varepsilon \frac{\partial}{\partial x}(\rho Y(1-Y)u_r) &= 0 \quad (1.2)\\
&\frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho (u)^2 + p) &= 0 \quad (1.3)
\end{aligned} \tag{1}$$

This system describes a two-phase medium composed of two immiscible fluids $k = 1, 2$ where the pressures in the phases 1 and 2 are equal and given by barotropic state. To be more specific, ρ here denote the total density of the flow, u its velocity while Y is a mass fraction expressing the relative proportion of the mass of one of the two fluid over the total mass. For definitiveness, we will assume that this mass fraction is relative to the fluid 2 : $Y = Y_2$. Finally, the pressures $p_k, k = 1, 2$ in the two phase are given by barotropic state laws $p_k = p_k(\rho_k)$. The phase densities as well as the pressure p are then found by solving the system of equations expressing the equality of the pressures in the two phases as well as the saturation constraint $\alpha_1 + \alpha_2 = 1$ with $\alpha_k = \rho Y_k/\rho_k$ giving :

$$\begin{cases} p_1(\rho_1) = p_2(\rho_2) \\ \dfrac{(1-Y)}{\rho_1} + \dfrac{(Y)}{\rho_2} = \dfrac{1}{\rho} \end{cases}$$

The partial mass conservation equation (1.2) contains a diffusive term that expresses the fact that the velocities of the two phases are not exactly equal. They differs from the centre of mass velocity u by a relative velocity u_r. An asymptotic analysis[2] provides the following estimate for this term

$$u_r = (Y - \alpha)\frac{\partial p}{\partial x} \tag{2}$$

where α is the volume fraction. The second-order perturbation of this model presents some unusual features. In contrast with many dissipative systems, this model does not contain any regularizing term in the momentum equations but only contain a second-order perturbation in the partial mass equation. It can be shown that this second-order perturbation comes from the existence of relaxation phenomena between the two phases that drive the system toward mechanical equilibrium and that system (1) results from a rigorous Chapman-Enskog asymptotic analysis of non-equilibrium two phase models.[2]

3. TRAVELING WAVE ANALYSIS

In this section, we will use this model to study the possible regularizing effect in two-phase models of second-order perturbations of the form displayed in equation (1.2). Actually, since the model (1) does not contain a viscous regularisation in the momentum equation, one may wonder if the diffusive term in (1.2) is sufficient for a regularizing effect to occur. This is the main motivation of this work. Therefore, we now focus on the possible existence of a certain class of solutions of this system, namely the traveling waves solutions defined by

Definition $U(t,x) =^t (\rho, \rho Y, \rho u)$ is a traveling wave solution of (1) if

1. There exists a real s and a one-parameter function $\hat{U}(\xi)$ such that $U(t,x) = \hat{U}(x - st)$
2. There exist two state vectors U_L and U_R such that

$$\begin{cases} \lim_{\xi\to-\infty} \hat{U}(\xi) = U_L & (3.1) \\ \lim_{\xi\to-\infty} \hat{U}'(\xi) = 0 & (3.2) \\ \lim_{\xi\to+\infty} \hat{U}(\xi) = U_R & (3.3) \\ \lim_{\xi\to+\infty} \hat{U}'(\xi) = 0 & (3.4) \end{cases}$$

If such solutions exist, they are characterized by the differential system of degree 2 :

$$(F(\hat{U}))' - s\hat{U}' = (D(\hat{U})\hat{U}')' \tag{4}$$

Since system (1) is in conservative form, it can integrated once to yield a first-order system. Algebraic manipulations of the resulting first-order system of odes gives the following result[3] :

Lemma 0.1 *The differential system (4) associated with the right boundary conditions*

$$\begin{cases} \lim_{\xi\to+\infty} U(\xi) = U_R \\ \lim_{\xi\to+\infty} U'(\xi) = 0 \end{cases}$$

can be reduced to the following first order ode :

$$Y(1-Y)(\alpha - Y)z' \quad = \quad \tau \frac{z(M^2\tau_R - p_R - z)}{M(a_2^2 - a_1^1)} \quad (5) \tag{5}$$

where the scalar variable $z = p - p_R$ denote the pressure while p_R is the pressure of the downwind side of the travelling wave and where Y and α are functions of z defined by :

$$\begin{cases} Y(z) = Y_R + \dfrac{z(M^2\tau_R - p_R - z)}{M^2(a_2^2 - a_1^2)} & (6.1) \\ \\ \alpha(z) = \dfrac{a_2^2 Y(z)}{\tau(z)(p_R + z)} & (6.2) \end{cases} \qquad (6)$$

with M the mass flux and $\tau = 1/\rho$. The existence of traveling wave solutions thus reduces to the sudy of this ode. It is then easily seen that $z = 0$ and $z = z_L = M^2\tau_R - p_R$ are two equilibrium points. The linearization of (5) in the vicinity of these two points shows easily that $z = 0$ is a stable equilibrium while $z = z_L$ is unstable. To end this study, we then have to check if (5) can become singular and thus we are lead to study the function $Y(z)(1 - Y(z))(\alpha(z) - Y(z))$ for $z \in [0, z_L]$. This study[3] then reveals the existence of a critical mass flux M_{crit} splitting the set of solutions into two classes as follows :

Weak shock case If the mass flux verifies $M^2 < M_{crit}^2$ the ode (5) is never singular and therefore, it exists a unique $\mathcal{C}^1$ solution connecting the two equilibrium $z = 0$ and $z = z_L = M^2\tau_R - p_R$ and in consequence a viscous profile connecting the two states U_L and U_R.
These continuous profiles are displayed in figure 1 from ref[3] where they have been computed by a numerical integration of (5) as well as by a finite volume approximation of the original PDE system (1).

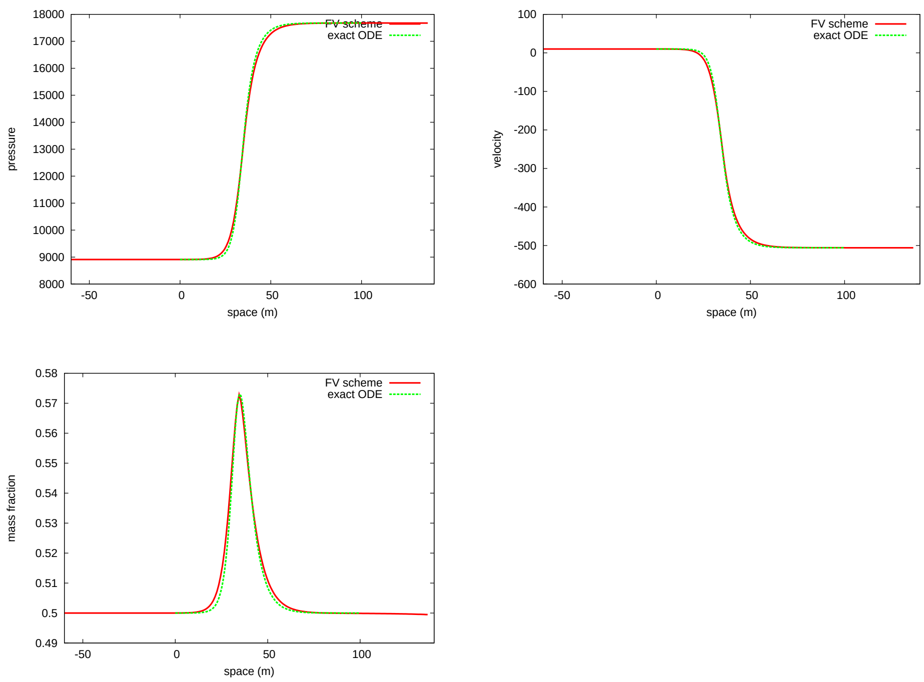

Figure 1: Profiles in the shock for the weak shock case, upper left : pressure, upper right : velocity, lower left : mass fraction

The second set of solutions is given by :

<u>**Strong shock case**</u> If the mass flux verifies $M^2 > M^2_{crit}$, there is no $\mathcal{C}^1$ viscous profile connecting the states U_L and U_R. The ode (5) is singular in z^*_L and z^*_R and there exist an infinite number of orbits connecting the two equilibria $z = 0$ and $z = z_L$. These orbits are composed of

- a $\mathcal{C}^1$ two-phase solution connecting the equilibrium points $z = 0$ and $z = z^*_R$ or equivalently U_R and U^*_R
- a discontinuous one-phase shock connecting the two states U^*_R and U^*_L
- a $\mathcal{C}^1$ two-phase solution connecting $z = z^*_L$ and the equilibrium $z = z_L$ or equivalently U^*_L and U_L

These discontinuous solutions again from[3] are displayed in figure 2

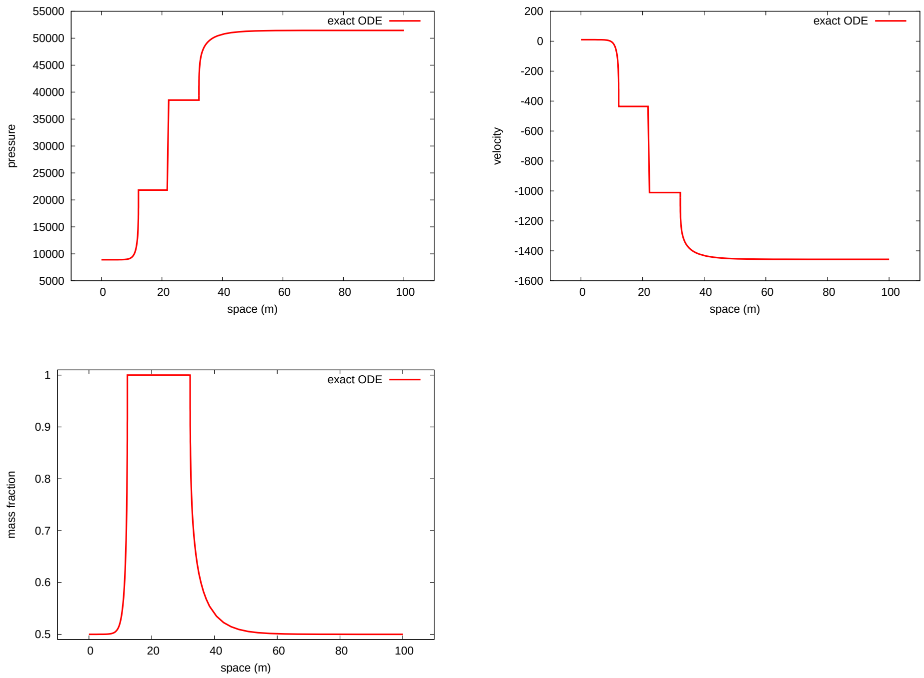

Figure 2: Profiles in the shock for the strong shock case obtained from numerical integration of (5) upper left : pressure, upper right : velocity, lower left : mass fraction

4. CONCLUSIONS

In this work, we have investigated in a simple model of two phase flow, a particular structure of dissipative tensor that differs from the usual viscous tensors constructed from a Navier-Stokes analogy.
Traveling wave analysis for this model has been performed. This analysis has revealed that despite the zero mass fraction jump implied by the Rankine-Hugoniot relations, the mass fraction is not constant in the shock region. Moreover, this analysis has also shown the existence of a critical speed of the waves above which no $\mathscr{C}^1$ solutions exist. However above this critical speed, we have shown the possibility to construct traveling solutions involving single phase shocks. This analysis thus reveals interesting features that cannot *a priori* be deduced from the non-dissipative

system. The same type of analysis performed with more complex two-phase models is currently under investigation and show a similar behavior.[4]

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