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APPLICATION OF THE MULTIGRID METHOD FOR CALCULATION DIFFUSION PROCESSES

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Abstract. The new effective algorithm of a solution of parabolic equations on the basis of a multigrid method which keeps in itself virtues of the implicit scheme - a stability and an exactitude Is offered, and thus allows to reduce essentially volume of arithmetical work on each temporary stratum. Theoretical and numerical research of stability and an exactitude of the created two-grid method on an example of modelling problems for one-dimensional and two-dimensional heat conduction equations with constants and variable factors is carried out. On an example of an one-dimensional modelling problem the absolute stability of the offered method is theoretically proved. On an example of a two-dimensional modelling problem the exactitude of a method is theoretically investigated. The solution received by means of the offered method is proved, that for a modelling problem, differs from a solution under the implicit scheme on a detailed grid very little. Calculations of one-dimensional and two-dimensional modelling problems including with explosive factors have shown a good exactitude of the offered method.

1. INTRODUCTION

Numerical modeling of many problems of mathematical physics is impossible without taking into account diffusion processes. Diffusion processes consideration requires the solution of parabolic equations. Use of explicit schemes for the approximation of parabolic equations leads to tough restriction in time step for preservation of stability[1,2] and therefore to very long calculations time. Application of implicit schemes removes this restriction, but it requires solving large systems of linear algebraic equations, which renders using such schemes unexpedient. The use of classical multigrid methods[3] may in some cases also involve high computational complexity. As a result, they may offer no advantage over explicit solvers. Therefore the development of new algorithms of parabolic equations solution should be done.

This report represents the new effective algorithm based on multigrid method use. Theoretical and numerical research of stability and exactitude of the created two-grid method on an example of modelling problems for one-dimensional and two-dimensional heat conduction equations with constants and variable factors is carried out. It is shown, that the offered algorithm keeps in itself virtues of the implicit scheme - a stability and an exactitude, and besides allows to reduce essentially volume of arithmetical work on each temporary stratum in comparison with use of the implicit scheme.

2. EXPOSITION OF ALGORITHM

The presentation is restricted to the construction and analysis of a two-grid algorithm for diffusion-type equations. As an example, we consider initial–boundary value problem (1) for one- and two-dimensional heat equations.

$$\begin{gathered}\rho C_v \frac{\partial T}{\partial t} = div(\kappa gradT) + f, \quad x \in G, \\ T(x,t) = g(x,t), \quad x \in \gamma, \\ T(x,0) = T_0(x).\end{gathered} \tag{1}$$

In (1), T is the temperature in point x at t, C_v is the coefficient capacity of heat, and κ is the heat conduction, ρ - density, f density of heat source, γ - boundary of calculation area, $g(x,t), T_0(x)$ - the set functions

To approximate problem (1) on a rectangular computational domain $G = \{0 < x < l_1,\ 0 < y < l_2,\ 0 < t \le T_l\}$, we use fully implicit finite-difference scheme (2):

$$\begin{gathered}(C_v\rho)_{ij} \frac{u_{ij}^{n+1} - u_{ij}^{n}}{\tau} = \kappa_{i+0.5,j} \frac{u_{i+1j}^{n+1} - u_{ij}^{n+1}}{h_x^2} - \kappa_{i-0.5,j} \frac{u_{ij}^{n+1} - u_{i-1j}^{n+1}}{h_x^2} + \kappa_{ij+0.5} \frac{u_{ij+1}^{n+1} - u_{ij}^{n+1}}{h_y^2} - \\ \kappa_{ij-0.5} \frac{u_{ij}^{n+1} - u_{ij-1}^{n+1}}{h_y^2} + \Phi_{ij}, \quad 0 < i < N_1,\ 0 < j < N_2,\end{gathered} \tag{2}$$

$$\begin{gathered}u_{ij}^0 = T_0(x_i, y_j), \quad 0 \le i \le N_1,\ 0 \le j \le N_2, \\ u_{0j}^n = u_1\left(t_n, y_j\right), u_{N_1j}^n = u_2\left(t_n, y_j\right), \\ u_{i0}^n = u_3\left(t_n, x_i\right), u_{iN_2}^n = u_4\left(t_n, x_i\right), \\ 0 < i < N_1,\ 0 < j < N_2.\end{gathered}$$

Where h_x, h_y – constant steps of a grid on x, y, and τ - step on t. Difference scheme (2) represents the linear algebraic equations system (3) concerning unknown values of the solution on n+1 time level.

$$A_h u^{n+1} = f_h \ . \tag{3}$$

In the method proposed here, the values of the grid function on the next time level are determined by means of the following algorithm.

At the first stage, one or several smoothing simple iterations for the equation (2) or (3) are done, calculations are made by the formula (4).

$$\overset{s+1}{\overset{n+1}{u}}_{ij} = \sigma\left(\frac{(C_v\rho)_{ij}\overset{n}{u}_{ij}}{\tau} + \frac{\kappa_{i+0.5,j}\overset{s}{\overset{n+1}{u}}_{i+1j} + \kappa_{i-0.5,j}\overset{s}{\overset{n+1}{u}}_{i-1j}}{h_x^2} + \frac{\kappa_{i,j+0.5}\overset{s}{\overset{n+1}{u}}_{ij+1} + \kappa_{i,j-0.5}\overset{s}{\overset{n+1}{u}}_{ij-1}}{h_y^2} + \Phi_{ij}\right)\times$$
$$\left(\frac{1}{\frac{(C_v\rho)_{ij}}{\tau} + \frac{\kappa_{i+0.5,j}+\kappa_{i-0.5,j}}{h_x^2} + \frac{\kappa_{i,j+0.5}+\kappa_{i,j-0.5}}{h_y^2}}\right) + (1-\sigma)\overset{s}{\overset{n+1}{u}}_{ij}, \tag{4}$$

Where $i = 1,2,...,N_1-1,\ j = 1,2,...,N_2-1$, σ - a weight factor, $0 < \sigma \le 1$, $\overset{0}{\overset{n+1}{u}}_{ij} = u_{ij}^n$. The received grid function we shall designate $u_{ij}^{sm.}$. Then residual is calculated by formula:

$$r = A_h u^{sm.} - f_h\,. \tag{5}$$

At the second stage – the stage of the projection or restriction of the residual on coarse grid the calculation are made by the formulas:

$$R_{lp} = r_{2i_1,2j_1}, \quad where \quad l = i_1 = 1,2,...,N_1/2-1, \quad p = j_1 = 1,2,...,N_2/2-1\,. \tag{6}$$

At the third stage the equation is solved for the correction on the coarse grid. In the considered two-dimensional case it is equation

$$(C_v\rho)_{lp}\frac{\Delta_{lp}}{\tau} - \kappa_{l+0.5,p}\frac{\Delta_{l+1,p}-\Delta_{lp}}{H_x^{\ 2}} + \kappa_{l-0.5,p}\frac{\Delta_{lp}-\Delta_{l-1,p}}{H_x^{\ 2}} -$$
$$\kappa_{l,p+0.5}\frac{\Delta_{l,p+1}-\Delta_{lp}}{H_y^{\ 2}} + \kappa_{l,p-0.5}\frac{\Delta_{lp}-\Delta_{l,p-1}}{H_y^{\ 2}} = R_{lp}, \tag{7}$$
$$\Delta_{l0} = \Delta_{l,N_2/2} = \Delta_{0p} = \Delta_{N_1/2,p} = 0,$$

$$l = 1,2,...,N_1/2-1, \quad p = 1,2,...,N_2/2-1,$$

where $H_x = 2h_x$, $H_y = 2h_y$.

At the fourth stage the coarse grid correction is interpolated to the fine grid. We use formulas (8) to perform 4-point face-centered and 16-point cell-centered (cubic) interpolation from the coarse grid.

$$\delta_{ij} = \begin{cases} \Delta_{lp}, \quad i=2l, j=2p \quad \textit{for even} \quad i \textit{ and } j, \\ \frac{9}{16}\left(\Delta_{lp}+\Delta_{l+1,p}\right)-\frac{1}{16}\left(\Delta_{l-1,p}+\Delta_{l+2,p}\right), \quad i=2l+1, j=2p \quad \textit{for odd } i \textit{ and even } j, \\ \frac{9}{16}\left(\Delta_{lp}+\Delta_{l,p+1}\right)-\frac{1}{16}\left(\Delta_{l,p-1}+\Delta_{l,p+2}\right), \quad i=2l, j=2p+1, \quad \textit{for odd } j \textit{ and even } i, \\ \frac{81}{256}\left(\Delta_{lp}+\Delta_{l+1,p}+\Delta_{l,p+1}+\Delta_{l+1,p+1}\right)+ \\ \frac{1}{256}\left(\Delta_{l-1,p-1}+\Delta_{l-1,p+2}+\Delta_{l+2,p+2}+\Delta_{l+2,p-1}\right)- \\ \frac{9}{256}\left(\Delta_{l-1,p}+\Delta_{l-1,p+1}+\Delta_{l,p+2}+\Delta_{l+1,p+2}+\Delta_{l+2,p+1}+\Delta_{l+2,p}+\Delta_{l+1,p-1}+\Delta_{l,p-1}\right), \\ \qquad i=2l+1, j=2p+1, \end{cases} \tag{8}$$

where $i=1,2,...,N_1-1,\ j=1,2,...,N_2-1$. We shall notice, that $\delta_{i0}=\delta_{iN_2}=0,\ \delta_{0j}=\delta_{N_1j}=0$.

At a final fifth stage, calculation of grid function on the next time level is made by the formula:

$$u_{ij}=u_{ij}^{sm.}-\delta_{ij}\,. \tag{9}$$

Thus, only one iteration of the two-grid cycle is made. In spite of the fact that the system of linear equations thus remains unsolved completely, such scheme has stability and accuracy inherent in the implicit scheme. This is demonstrated below both theoretically and numerically for some modeling problems. Moreover, when the number of fine grid points is sufficiently large, the computational costs are lower in the proposed method as compared to the implicit algorithm on the fine grid, because the solution of coarse grid correction equation (7) requires essentially less arithmetic work than the solution of the equations system for the implicit scheme by the fine.

3. TEORETICAL ANALYSIS OF STABILITY OF THE METHOD

We use Fourier analysis[4,5] to inverstigate stability of the two-grid method with respect to perturbations of the initial conditions for the model problem. We consider the initial–boundary value problem for the one-dimensional heat equation with unit coefficients on the unit interval with zero boundary conditions. Suppose that N is an even number. We shall assume, that one smoothing iteration is done. The implicit scheme on the fine grid is given by (10).

$$\begin{gathered} \frac{u_i^{im}-\check{u}_i}{\tau}=\frac{u_{i-1}^{im}-2u_i^{im}+u_{i+1}^{im}}{h^2}+\Phi_i, \quad i=1,2,...,N-1 \\ u_0=u_N=0 \\ u_i^0=(T_0)_i \end{gathered} \tag{10}$$

where u_i^{im} - a solution of a heat conduction equation under the implicit scheme on n+1 time level, $h=1/N$, $(T_0)_i$ - the set grid function.

We represent $\overset{\vee}{u}_i$ the solution on the n-th level as Fourier series expansion.

$$\overset{\vee}{u}_i = \sum_{k=1}^{N-1} a_k \sin k\pi x_i \sqrt{2}$$

In paper[6] Fourier expansion of the solution for n+1 time level determined by means of the proposed method.:

$$u_i = \sum_{\substack{k=1 \\ k \neq \frac{N}{2}}}^{N-1} \left[\left(q_{sm.}^k - Q_{cor.}^k q_{res.}^k \frac{q_k' + 1}{2} \right) a_k + Q_{cor.}^k q_{res.}^{N-k} \frac{q_k' + 1}{2} a_{N-k} \right] \sqrt{2} \sin k\pi x_i + \tag{11}$$
$$q_{sm.}^{N/2} a_{\frac{N}{2}} \sqrt{2} \sin\left(\frac{N}{2} \pi x_i \right),$$

where

$$q_{sm.}^k = 1 + \frac{\sigma R}{1+R}(q_k - 1), \quad q_{res.}^k = \frac{1}{\tau}\left\{ q_{sm.}^k \left[1 + R(1 - q_k) \right] - 1 \right\},$$
$$Q_{cor.}^k = \frac{\tau}{1 + 0.5R\left(1 - q_k^2\right)}, \tag{12}$$
$$q_k = \cos\frac{k\pi}{N} = \cos k\pi h, \quad q_k' = q_k[1 + 0.5(1 - q_k^2)], \quad R = \frac{2\tau}{h^2}.$$

Let's notice, in case one-dimensional problem an interpolation at 4 stage is made under formulas:

$$\delta_i = \begin{cases} \Delta_l, \quad l = \dfrac{i}{2} \quad for \quad i \quad even, \\ \dfrac{9}{16}\left(\Delta_l + \Delta_{l+1}\right) - \dfrac{1}{16}\left(\Delta_{l-1} + \Delta_{l+2}\right), \quad l = \dfrac{i-1}{2}, \quad for \quad i \quad odd, \end{cases}$$

where $i = 1, \ldots N-1$.

Let's prove now the absolute stability of the proposed method by the initial data in some special norm at $\sigma = 0.5$. We define the linear subspace H^k as the span of the Fourier modes $\sqrt{2}\sin k\pi x_i$ u $\sqrt{2}\sin\left(N-k\right)\pi x_i$, and $k = 1, 2, \ldots N/2 - 1$. Calculations by the proposed two-grid method, from formulas (11) and equalities $Q_{погр}^k = Q_{погр}^{N-k}, q_{N-k}' = -q_k'$ follows transform vector

$$\mathbf{x}^k = a_k \sqrt{2} \sin k\pi x_i + a_{N-k} \sqrt{2} \sin\left(N-k\right)\pi x_i \in H^k$$

into vector

$$y^k = A_k x^k \in H^k,$$

where

$$A_k = \begin{pmatrix} q^k_{сгл.} - Q^k_{попр.} q^k_{нев.} \dfrac{q'_k + 1}{2} & Q^k_{попр.} q^{N-k}_{нев.} \dfrac{q'_k + 1}{2} \\ Q^k_{попр.} q^k_{нев.} \dfrac{1 - q'_k}{2} & q^{N-k}_{сгл.} - Q^k_{попр.} q^{N-k}_{нев.} \dfrac{1 - q'_k}{2} \end{pmatrix}, \quad k = 1,2,...N/2-1.$$

It is proved[6] that the eigenvalues of the matrix A_k satisfy inequalities :

$$\lambda^k_1 \neq \lambda^k_{2,} \quad \left|\lambda^k_1\right| \leq 1 \quad u \quad \left|\lambda^k_2\right| \leq 1. \tag{13}$$

Define the norm $\| \ \|_1$ for all vectors in the space of grid function as:

$$\left\| \overset{\vee}{u} \right\|^2_1 = \sum_{k=1}^{N/2-1} \left(\alpha^k_1\right)^2 + \left(\alpha^k_2\right)^2 + a^2_{N/2},$$

where α^k_1, α^k_2 and $a_{N/2}$ are the coordinates of the vector $\overset{\vee}{u}$ in the basis $\overline{\mathbf{e}}^k_1, \overline{\mathbf{e}}^k_2$, $\sqrt{2}\sin\left(\dfrac{N}{2}\pi x_i\right)$, $k = 1,2,...N/2-1$; $\overline{\mathbf{e}}^k_1, \overline{\mathbf{e}}^k_2$ - are the eigenvectors associated with the aigenvalues λ^k_1, λ^k_2.

Using (13) and $\left|q^{km}_{сгл}\right| \leq 1$, we obtain

$$\|u\|_1 \leq \left\| \overset{\vee}{u} \right\|_1,$$

which proves absolute stability of the method with respect to perturbations of the initial conditions in norm $\| \ \|_1$.

It can be shown that the norms $\| \ \|_1$ and $\| \ \|_{L_2}$ are equivalent[6].

Proof of stability of the method by the right part is done similarly[6]. Thus, it is proved that the algorithm is absolutely stable with respect to perturbations of both initial conditions and right-hand side of the model problem considered here, and the estimation:

$$\left\|u^{n+1}\right\|_1 \leq \left\|u^0\right\|_1 + \tau Q_1 \sum_{j=0}^{n} \left\|\Phi^j\right\|_2 .$$

is valid. Where $Q_1 = const$ is independent of h, τ. The norm $\| \ \|_2$ is definet by analogy with $\| \ \|_1$.

4. ESTIMATION OF THE SOLUTION ERROR

As a model example, we consider the two-dimensional initial–boundary value problem for the heat equation with unit coefficients on the unit square with zero boundary conditions.:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad 0 < x < 1, 0 < y < 1, 0 \le t \le T,$$
$$u(x, y, 0) = T_0(x, y), \qquad 0 \le x \le 1, 0 \le y \le 1,$$
$$u(0, y, t) = 0, u(1, y, t) = 0, \quad 0 \le y \le 1, 0 \le t \le T,$$
$$u(x, 0, t) = 0, u(x, 1, t) = 0, \quad 0 \le x \le 1, 0 \le t \le T.$$

We propose that $T_0(x, y)$ is analytic function

The implicit scheme on the fine grid is :

$$\frac{u_{ij}^{im} - \check{u}_{ij}}{\tau} = \frac{u_{i-1,j}^{im} - 2u_{i,j}^{im} + u_{i+1,j}^{im}}{h^2} + \frac{u_{i,j-1}^{im} - 2u_{i,j}^{im} + u_{i,j+1}^{im}}{h^2},$$
$$i = 1, 2, ..., N-1, \quad j = 1, 2, ..., N-1, \tag{14}$$
$$u_{i0}^{im} = u_{i,N}^{im} = u_{0j}^{im} = u_{N,j}^{im} = 0,$$
$$u_{ij}^{0} = (T_0)_{ij}, \quad i = 0, 1, ..., N, j = 0, 1, ..., N.$$

where $h = 1/N$, $(T_0)_{ij}$ the grid function approximating, $T_0(x, y)$. We propose that N - is an even.

We propose that one smoothing iteration $\sigma = 0.5$ is done. We represent the solution on the nth time level as Fourier series expansion:

$$u_{ij}^{n} = \sum_{k=1}^{N-1} \sum_{m=1}^{N-1} a_{km} 2 \sin k\pi x_i \sin m\pi y_j,$$
$$i = 0, 1, ..., N, \tag{15}$$
$$j = 0, 1, ..., N.$$

We use the reasoning's similar to the ones presented above. We obtain that Fourier expansion of the solution by the means of the proposed method on N+1 time level is given by formula (15).

We shall receive(16),(17) after substitution u_{ij}^{n}, certain in (15), in a right member of the formula (4) at $C_v \rho \equiv 1, \kappa_{ij} \equiv 1, \Phi_{ij} \equiv 0, s = 0$, $h_x = h_y = h$ as a result of the simple transformations considering, the formula of the sum of sine of angles, after realization of smoothing stage.

$$u_{ij}^{sm.} = \sum_{k=1}^{N-1} \sum_{m=1}^{N-1} q_{sm}^{km} a_{km} 2 \sin k\pi x_i \sin m\pi y_j, \tag{16}$$

where

$$q_{sm}^{km} = 1 + \frac{0.5R(q_k + q_m - 2)}{1 + 2R}, \tag{17}$$

q_k, q_k' are rating in (12), $R = 2\tau / h^2$.

After substitution in (14) instead of u_{ij}^{im} the function u_{ij}^{sm} certain in (16), and instead of $\check{u}_{ij}$ function u_{ij}^{n}, certain in (15), we shall receive, that the Fourier-series expansion of a discrepancy on a detailed grid looks like:

$$r_{ij} = \sum_{k=1}^{N-1}\sum_{m=1}^{N-1} q_{res}^{km} a_{km} 2\sin k\pi x_i \sin m\pi y_j, \quad where \qquad (18)$$
$$q_{res}^{km} = \left\{ q_{sm}^{km}\left[1 + R(2 - q_k - q_m)\right] - 1 \right\} / \tau$$

After realization of the second stage - restrictions of a discrepancy on a rough grid, considering formulas $\sin(N-k)\pi x_{2i} = -\sin k\pi x_{2i}$ $\sin(0.5\pi N x_{2i}) = 0$, we shall receive, that the Fourier-series expansion of function R_{lp} looks like:

$$R_{lp} = \sum_{k=1}^{N/2-1}\sum_{m=1}^{N/2-1} (q_{нев}^{km} a_{km} - q_{нев}^{k,N-m} a_{k,N-m} - q_{нев}^{N-k,m} a_{N-k,m} + q_{нев}^{N-k,N-m} a_{N-k,N-m}) \times \qquad (19)$$
$$2\sin k\pi x_l \sin m\pi y_p,$$

where

$$x_l = x_{2i}, \quad y_p = y_{2j},$$
$$l = 1,\dots,N/2-1, \quad p = 1,\dots,N/2-1,$$
$$i = 1,\dots,N/2-1, \quad j = 1,\dots,N/2-1.$$

Let's present Δ_{lp} - solution of the equation for correction on a coarse grid

$$\frac{\Delta_{lp}}{\tau} - \frac{\Delta_{l+1p} - \Delta_{lp}}{H^2} + \frac{\Delta_{lp} - \Delta_{l-1p}}{H^2} - \frac{\Delta_{lp+1} - \Delta_{lp}}{H^2} + \frac{\Delta_{lp} - \Delta_{lp-1}}{H^2} = R_{lp}, \qquad (20)$$
$$\Delta_{0p} = \Delta_{N/2,p} = 0, \ \Delta_{l0} = \Delta_{l,N/2} = 0,$$

where $l = 1,\dots,N/2-1, p = 1,\dots,N/2-1, H = 2h$, in Fourier series

$$\Delta_{lp} = \sum_{k=1}^{N/2-1}\sum_{m=1}^{N/2-1} \tilde{a}_{km} 2\sin k\pi x_l \sin m\pi y_p \ . \qquad (21)$$

Let's substitute (19) and (21) in (20), in view of the formula for the sum of sine of angles after simple transformations we shall receive:

$$\Delta_{lp} = \sum_{k=1}^{N/2-1}\sum_{m=1}^{N/2-1} Q_{cor}^{km} (q_{res}^{km} a_{km} - q_{res}^{k,N-m} a_{k,N-m} - q_{res}^{N-k,m} a_{N-k,m} + q_{res}^{N-k,N-m} a_{N-k,N-m}) \times$$
$$2\sin k\pi x_l \sin m\pi y_p, \qquad (22)$$
$$where \quad Q_{cor}^{km} = \frac{\tau}{1 + 0.5R(2 - q_k^2 - q_m^2)}$$

Let's make interpolation Δ_{lp} on a fine grid in two stages. First we shall make interpolation on a grid $\{(ih, pH), i = 1,\dots,N-1, p = 1,\dots N/2-1\}$ under formulas

$$\tilde{\Delta}_{ip} = \begin{cases} \Delta_{lp}, \quad l = \dfrac{i}{2} \quad for \quad i \quad even, \\ \dfrac{9}{16}\left(\Delta_{lp} + \Delta_{l+1,p}\right) - \dfrac{1}{16}\left(\Delta_{l-1,p} + \Delta_{l+2,p}\right), \quad l = \dfrac{i-1}{2}, \quad for \quad i \quad odd. \end{cases} \tag{23}$$

Then we interpolate $\tilde{\Delta}_{ip}$ on fine grid under the formulas similar (23). It is easy to show, that as result of interpolation realization in such a way we shall receive interpolation under formulas (8). Arguing similarly[6] on each of two stages of interpolation, after simple transformations we shall receive, that the Fourier-series expansion of grid function δ_{ij} looks like:

$$\delta_{ij} = \sum_{k \neq N/2} \sum_{m \neq N/2} 0.25(q'_k + 1)(q'_m + 1) Q_{cor}^{km} (q_{res}^{km} a_{km} - q_{res}^{k,N-m} a_{k,N-m} - q_{res}^{N-k,m} a_{N-k,m} + q_{res}^{N-k,N-m} a_{km}) \times$$
$$2 \sin k\pi x_i \sin m\pi y_j$$

After realization of the fifth stage when evaluations are made under formulas (9), we shall receive

$$u_{ij} = \sum_{k \neq N/2} \sum_{m \neq N/2} (b_{km}^1 a_{km} + b_{km}^2 a_{k,N-m} + b_{km}^3 a_{N-k,m} - b_{km}^4 a_{N-k,N-m}) 2 \sin k\pi x_i \sin m\pi y_j +$$
$$\sum_{m=1}^{N-1} q_{sm}^{N/2,m} a_{N/2,m} 2 \sin \frac{N}{2} \pi x_i \sin m\pi y_j + \sum_{k \neq N/2} a_{k,N/2} q_{sm}^{k,N/2} 2 \sin k\pi x_i \sin \frac{N}{2} \pi y_j, \tag{24}$$

where

$$b_{km}^1 = q_{sm}^{km} - Q_{cor}^{km} 0.25(q'_k + 1)(q'_m + 1) q_{res}^{km}, \quad b_{km}^2 = Q_{cor}^{km} 0.25(q'_k + 1)(q'_m + 1) q_{res}^{k,N-m},$$
$$b_{km}^3 = Q_{cor}^{km} 0.25(q'_k + 1)(q'_m + 1) q_{res}^{N-k,m}, \quad b_{km}^4 = Q_{cor}^{km} 0.25(q'_k + 1)(q'_m + 1) q_{res}^{N-k,N-m}, \tag{25}$$

$q_{sm}^{km}, q_{res}^{km}, Q_{cor}^{km}$ are certain in (17), (18), (22), q_k , q'_k - in (12), $R = \dfrac{2\tau}{h^2}$.

So, in the formula (24) the Fourier-series expansion of a solution by means of the offered method on a new temporary stratum is reduced.

For research of an exactitude of the solution received by means of the offered method, all over again we shall make an estimation of an error of approximation of the implicit scheme on a detailed grid (14) on the solution received by means of the offered method. We shall substitute u_{ij} certain in (24), in the formula (14) instead of $u_{ij}^{н}$ and instead of $\check{u}_{ij}$ we shall substitute u_{ij}^{n}, certain in (15). The residual φ_{ij} looks like:

$$\varphi_{ij} = \sum_{k \neq N/2} \sum_{m \neq N/2} (r_{km}^1 a_{km} + r_{km}^2 a_{k,N-m} + r_{km}^3 a_{N-k,m} - r_{km}^4 a_{N-k,N-m}) 2 \sin k\pi x_i \sin m\pi y_j +$$
$$\sum_{m=1}^{N-1} r_m^5 a_{N/2,m} 2 \sin 0.5N\pi x_i \sin m\pi y_j + \sum_{k \neq N/2} r_k^6 a_{k,N/2} 2 \sin k\pi x_i \sin 0.5N\pi y_j,$$

where

$$r_{km}^{1}=\frac{b_{km}^{1}-1}{\tau}+\frac{2b_{km}^{1}(1-q_{k}+1-q_{m})}{h^{2}},\quad r_{km}^{2,3,4}=b_{km}^{2,3,4}\left(\frac{1}{\tau}+\frac{2(1-q_{k}+1-q_{m})}{h^{2}}\right),$$
$$r_{m}^{5}=\frac{q_{sm}^{N/2,m}-1}{\tau}+\frac{2q_{sm}^{N/2,m}(2-q_{m})}{h^{2}},\quad r_{k}^{6}=\frac{q_{sm}^{k,N/2}-1}{\tau}+\frac{2q_{sm}^{k,N/2}(2-q_{k})}{h^{2}}. \quad (26)$$

We use the reasoning's similar to the ones presented above. We obtain that Fourier expansion of the solution by the means of the proposed method on N+1 time level is given by formula:

$$\|\varphi\|_{L_2}\le\|\varphi^{1}\|_{L_2}+\|\varphi^{2}\|_{L_2}+\|\varphi^{3}\|_{L_2}+\|\varphi^{4}\|_{L_2}, \quad (27)$$

where the terms on the right-hand side are defined in:

$$\|\varphi^{1}\|_{L_2}^{2}=\sum_{k\ne N/2}\sum_{m\ne N/2}(a_{km})^{2}\left(r_{km}^{1}\right)^{2},$$
$$\|\varphi^{2}\|_{L_2}^{2}=\sum_{k\ne N/2}\sum_{m\ne N/2}(a_{km})^{2}\left(r_{k,N-m}^{2}\right)^{2},$$
$$\|\varphi^{3}\|_{L_2}^{2}=\sum_{k\ne N/2}\sum_{m\ne N/2}(a_{km})^{2}\left(r_{N-k,m}^{3}\right)^{2}, \quad (28)$$
$$\|\varphi^{4}\|_{L_2}^{2}=\sum_{k\ne N/2}\sum_{m\ne N/2}(a_{km})^{2}\left(r_{N-k,N-m}^{4}\right)^{2}+\sum_{m=1}^{N-1}\left(a_{N/2,m}\right)^{2}\left(r_{m}^{5}\right)^{2}+\sum_{k\ne N/2}\left(a_{k,N/2}\right)^{2}\left(r_{k}^{6}\right)^{2}$$

Suppose that condition

$$\tau=h^{\beta}\text{, где } 0<\beta<2.$$

is satisfied.

We assume that the solution on the previos time step u_{ij}^{n} has $2p$ bounded finite-difference derivatives with respect to both coordinates.

Let's make the estimation of the first term in (27). We consider 4 subdomains of the indexes change area k, m, represented on fig. 1, denote by $\Omega=\Omega_1\cup\Omega_2\cup\Omega_3\cup\Omega_4$.

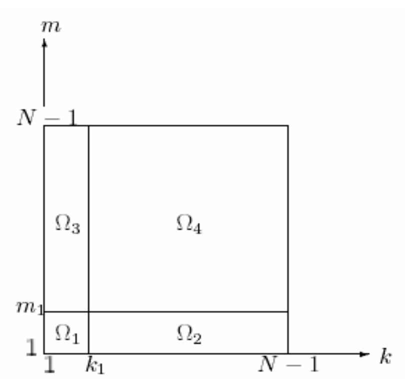

Figure 1. Partition of the domain of indices into subdomains: $k_1=m_1=\left[N^{\beta\delta}\right]$, $0<\delta<1/7$, $\left[N^{\beta\delta}\right]$ -is the integer part of $N^{\beta\delta}$

Estimates for $\left|r_{km}^1\right|$ and $\left|a_{km}\right|$ are as follows.

Inequations $k\pi h << 1$, $m\pi h << 1$, $k^2\pi^2\tau << 1$, $m^2\pi^2\tau << 1$ is true in subdomain Ω_1. We have:

$$\left|r_{km}^1\right| \le c_1 h^2 h^{\beta(1-6\delta)}, where \quad c_1 = \pi^6/3 = const., \tag{29}$$

Consideration that $0 < 1 - q_k < 2$, $0 < 0.5(1+q_k') < 1$, $0 < (1+q_k') < 9(1+q_k)/8$, one can proved $\left|b_{km}^1\right| < 17/16$, where b_{km}^1 from (25). Take account of (26), we have (30) in subdomain $\Omega_2 \cup \Omega_3 \cup \Omega_4$.

$$\left|r_{km}^1\right| \le c_2(1/h^2), \text{ where } c_2 = 4\frac{3}{16} = \text{const.} \tag{30}$$

Similarly[6]

in subdomain $\Omega_1 : \left|a_{km}\right| \le g_1 = const$, (31)

in subdomain $\Omega_2 : \left|a_{km}\right| \le \dfrac{g_2}{k^{2p}}$, $\quad g_2 = const$, (32)

in subdomain $\Omega_3 : \left|a_{km}\right| \le \dfrac{g_3}{m^{2p}}$, $\quad g_3 = const$, (33)

in subdomain Ω_4: both (32) and (33) are valid.

We represent $\left\|\varphi_{ij}^1\right\|_{L_2}^2$ (from (28)) in the form of four items sum respective to change of indexes of summation in the subdomains Ω_1, Ω_2, Ω_3, Ω_4. It is proved that inequalities (35) are valid.

$$\left\|\varphi_{ij}^1\right\|_{L_2}^2 = A_{11} + A_{12} + A_{13} + A_{14}, \tag{34}$$

where

$$\begin{aligned}
A_{11} &= \sum_{(k,m)\in\Omega_1} \left(a_{km}\right)^2 \left(r_{km}^1\right)^2 \le A_{11}' = c_{11} h^4 h^{2\beta(1-7\delta)}, \\
A_{12} &= \sum_{(k,m)\in\Omega_2, k\ne N/2} \left(a_{km}\right)^2 \left(r_{km}^1\right)^2 \le A_{12}' = c_{12} h^4 h^{2p2\beta\delta-\beta\delta-9}, \\
A_{13} &= \sum_{(k,m)\in\Omega_3, m\ne N/2} \left(a_{km}\right)^2 \left(r_{km}^1\right)^2 \le A_{13}' = c_{13} h^4 h^{2p2\beta\delta-\beta\delta-9}, \\
A_{14} &= \sum_{(k,m)\in\Omega_4, k\ne N/2, m\ne N/2} \left(a_{km}\right)^2 \left(r_{km}^1\right)^2 \le A_{14}' = c_{14} h^4 h^{2p2\beta\delta-10}, \\
& c_{1n} = const,\ n = 1,2,3,4.
\end{aligned} \tag{35}$$

For rather large values $2p$ when the inequality $2p2\beta\delta - 6 > 4\beta(1-3.5\delta)$ is true, equalities $A_{14}' = o(A_{11}')$, $A_{12}' = o(A_{11}')$ and $A_{13}' = o(A_{11}')$ are valid. From inequalities (35) it follows that for sufficiently large values $2p$ the estimate (36) is true.

$$\left\|\varphi^1\right\|_{L_2} \le \bar{c}_1 h^2 h^{\beta(1-7\delta)},\ \bar{c}_1 = const. \tag{36}$$

It is similarly proved that for sufficiently large values $2p$, estimates (37) are true

$$\left\|\varphi^2\right\|_{L_2} \le \bar{c}_2 h^2 h^{\beta(1-7\delta)},\quad \left\|\varphi^3\right\|_{L_2} \le \bar{c}_3 h^2 h^{\beta(1-7\delta)},\quad \left\|\varphi^4\right\|_{L_2} \le \bar{c}_4 h^2 h^{\beta(1-7\delta)} \tag{37}$$

$\bar{c}_n = const, n = 2,3,4$. Considering the triangle inequality (27) presented above, for sufficiently large values $2p$ we obtain that the inequality (38) is true.

$$\left\|\varphi\right\|_{L_2} \le ch^2 h^{\beta(1-7\delta)},\quad c = const. \tag{38}$$

The inequality (38) gives the truncation error estimate of the implicit scheme on the fine grid on the solution obtained by the proposed method.

Since the implicit scheme for the heat equation is stable in the L_2 norm[1], we have estimate (39) if $2p$ is sufficiently large.

$$\left\|u_{ij} - u_{ij}^{im}\right\|_{L_2} = O(h^2 h^{\beta(1-7\delta)}) = O(h^2 \tau^{(1-7\delta)}) = o(\tau + h^2) \tag{39}$$

Here, u_{ij}^{im} and u_{ij} are the solutions to the heat equation on the next time level obtained by applying the implicit scheme on the fine grid and the proposed method, respectively. As a result, we have the

Lemma 1. Suppose that the solution u_{ij}^n of the model problem has any number of bounded finite-difference derivatives with respect to both coordinates. Let $\tau = h^\beta$, where $0 < \beta < 2$. Then, $\left\|u_{ij} - u_{ij}^{\text{im}}\right\|_{L_2} \to 0$ as $h \to 0$ *and* $\tau \to 0$, and estimate (39) holds, where δ is any real number such that $0 < \delta < 1/7$

Equality (39) gives an estimation for the difference between the solutions to the model problem obtained by applying the proposed method and the implicit scheme on the fine grid.

Using triangle inequality $\left\|u_{ij}^T - u_{ij}^{n+1}\right\|_{L_2} \le \left\|u_{ij}^T - u_{ij}^{н}\right\|_{L_2} + \left\|u_{ij}^{n+1} - u_{ij}^{н}\right\|_{L_2}$, where u_{ij}^T is the projection of the solution of the modal problem to the fine grid, and using the fact than[1]

$$\left\|u_{ij}^T - u_{ij}^{н}\right\|_{L_2} = O(\tau + h^2) \tag{40}$$

we obtain:

$$\left\|u_{ij}^T - u_{ij}^{n+1}\right\|_{L_2} = O(\tau + h^2). \tag{41}$$

Equality (41) which estimates the accuracy of the solution obtained by the proposed method. Combining (39) with (40) we see that $\left\|u_{ij} - u_{ij}^{н}\right\|_{L_2} \to 0$ faster than $\left\|u_{ij}^T - u_{ij}^{н}\right\|_{L_2} \to 0$ as $h \to 0, \tau \to 0$.

Thus, it has been proved that the proposed method allows to obtain smooth solution of model two-dimensional problem with the same accuracy order as by pure implicit scheme on the fine grid, and very close to the solution under pure implicit scheme on the fine grid.

5. RESULTS OF NUMERICAL EXPERIMENTS

The proposed method was tested by computing nine model problems. We comput one-dimensional model problem, which has an exact solution

We solve problem for the one-dimensional heat equation with unit coefficients on the unit interval with boundary conditions of the first kind (Problem 1).

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2},\quad 0<x<1,\quad 0<t\le T,$$
$$u(x,0)=sin(\pi x),\quad 0\le x\le 1,$$
$$u(0,t)=0,\quad u(1,t)=0,\quad 0\le t\le T$$

where T=0.0679. One smoothing iteration was performed before switching to the coarser grid level., Solution of system of equations for deduction realize of sweep method[1,7]. Problem1 has an exact solution $u_T(x,t)=e^{-\pi^2 t}\sin(\pi x)$. Table 1 specifies the maximal value of the absolute error in percentage, and $K=\tau/h^2$.

	N = 100	N = 500	N = 1000
K = 1	0,039	0,00159	0,0005
K = 10	0,34	0,0139	0,0036
K = 100	3,25	0,136	0,034
K = 1000		1,3	0,34
K = 10000			3,25

Table 1.The maximum deviation from the exact solution, $\max_{x,t}\left|\frac{u_T(x_i,t)-u(x_i,t)}{u_T(x_i,t)}\right|$

The computed results demonstrate the stability and accuracy of the method.

We calculate two-dimensional modeling problem 2 having exact solution

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2},\quad 0<x<1,\quad 0<y<1,\quad 0<t\le T,$$
$$u(x,y,0)=sin(\pi x)sin(\pi y),\quad 0\le x\le 1,\quad 0\le y\le 1,$$
$$u(0,y,t)=0,\quad u(1,y,t)=0,\quad 0\le y\le 1,\quad 0\le t\le T,$$
$$u(x,0,t)=0,\quad u(x,1,t)=0,\quad 0\le x\le 1,\quad 0\le t\le T.$$

One smoothing iteration was performed before switching to the coarser grid level. and $\sigma = 0.5$. Solution of system of equations for correction realize of Seidel method[1,7]. We assume $h_x = h_y = 1/N$. Table 2 has the maximal value of absolute $\max_{i,j,t} \left| u_{ij}^T - u_{ij} \right|$ (the first number) and relative $\max_{i,j,t} \left| (u_{ij}^T - u_{ij}) / u_{ij}^T \right|$ (the second number) errors at the solution of problem 2, and $u_{ij}^T = e^{-2\pi^2 t} \sin(\pi x)\sin(\pi y)$, $t \in (0, 0.199]$

	N = 50		N = 100	
K = 1	3,3888 10^{-4}	1,757 10^{-2}	8,405 10^{-5}	4,3563 10^{-3}
K = 5	1,5687 10^{-3}	8,1337 10^{-2}	3,8644 10^{-4}	2,0029 10^{-2}
K = 10	3,1400 10^{-3}	0,1628	7,6655 10^{-4}	3,9730 10^{-2}
K = 20	6,3890 10^{-3}	0,3312	1,5338 10^{-3}	7,9495 10^{-2}
K = 100	3,5746 10^{-2}	1,8534	7,9761 10^{-3}	0,4134
K =1000			9,3930 10^{-2}	4,868

	N = 200		N=500	
K = 1	2,0971 10^{-5}	1,0868 10^{-3}		
K = 5	9,6259 10^{-5}	4,988 10^{-3}		
K = 10	1,9050 10^{-4}	9,8720 10^{-3}	3,0429 10^{-5}	1,5769 10^{-3}
K = 20	3,7943 10^{-4}	1,9662 10^{-2}		
K = 100	1,9116 10^{-3}	9,9070 10^{-2}	3,0186 10^{-4}	1,5644 10^{-2}
K =1000	2,1120 10^{-2}	1,0948	3,081 10^{-3}	0,1596

Table 2. Value $\max_{i,j,t} \left| u_{ij}^T - u_{ij} \right|$ and $\max_{i,j,t} \left| (u_{ij}^T - u_{ij}) / u_{ij}^T \right|$ obtained by solving Problem 2.

The results of the calculations confirm stability and accuracy of the method.

Table 3 provides with the maximal values of absolute (the first number) and relative (the second number) errors at the solution of a problem 2 with the use of the implicit scheme for heat conduction equation on the fine grid (in the third and fourth columns) and by the means of the algorithm proposed in the report (in first two columns).

As may be seen from table 3 these numbers differ from each other very slightly.

	N=100		Implicit scheme N = 100	
K = 1	8,405 10^{-5}	4,3563 10^{-3}	8,3891 10^{-5}	4,3480 10^{-3}
K = 5	3,8644 10^{-4}	2,0029 10^{-2}	3,88581 10^{-4}	19997 10^{-2}
K = 10	7,6655 10^{-4}	3,9730 10^{-2}	7,6532 10^{-4}	3,9666 10^{-2}
K = 20	1,5338 10^{-3}	7,9495 10^{-2}	1,5312 10^{-3}	7,93628 10^{-2}
K = 100	7,9761 10^{-3}	0,4134	7,947 10^{-3}	0,41243
K = 1000	9,3930 10^{-2}	4,868	9,3786 10^{-2}	4,86095

Table 3. The values of $\max\limits_{i,j,t}\left|u_{ij}^{T}-u_{ij}\right|$, $\max\limits_{i,j,t}\left|(u_{ij}^{T}-u_{ij})/u_{ij}^{T}\right|$, $\max\limits_{i,j,t}\left|u_{ij}^{T}-u_{ij}^{\text{н}}\right|$, $\max\limits_{i,j,t}\left|(u_{ij}^{T}-u_{ij}^{\text{н}})/u_{ij}^{T}\right|$, obtained by solving Problem 2.

We consider the initial–boundary value problem for the heat equation:

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}k\frac{\partial u}{\partial x}+\frac{\partial}{\partial y}k\frac{\partial u}{\partial y},\ 0<x<1,\ 0<y<1,\ 0<t\le T, \qquad (42)$$

$$u(x,y,0)=1+x+y,\ 0\le x\le 1,\ 0\le y\le 1,$$

$$u(0,y,t)=1+y,\, u(0,y,t)=2+y,\ 0\le y\le 1,\ \ 0\le t\le T,$$

$$u(x,0,t)=1+x,\ u(x,0,t)=2+x,\ \ 0\le x\le 1,\ \ 0\le t\le T,$$

where *k(x,y)* see below. Approximation of the heat conduction factor was carried out in the grid nodes. In difference scheme on the fine and coarse grids, half-sum of values in the nearest nodes was used. The problems were computed by the two-grid algorithm described above. One, two, or three smoothing iterations were performed before switching to the coarser grid level. Solution of system of equations for deduction realize of MICCG(0) method[8]. The solution obtained by the proposed method was compared with the solution obtained by using the implicit scheme on the fine grid.

In problems 3- 7 the heat conduction factor is discontinuous, and there is the factor jump. In problem 3

$$k=\begin{cases}100, x\in\left(0.25,0.75\right), y\in\left(0.25,0.75\right),\\ 1, othewise.\end{cases}$$

In problem 4

$$k=\begin{cases}100, x\in[0.25,0.75] y\in[0.25,0.75],\\ 1, othewise.\end{cases}$$

Tables 4, 5 show the relative error $\max\limits_{i,j}\left|(u_{ij}-u_{ij}^{\text{н}})/u_{ij}^{\text{н}}\right|$ in time $t=0.199$ in case if the approximation of heat conduction factor was carried out in the nodes of the coarse grid by the formula (43), which we call Approximation 1:

$$k'_{lm} = k(x_l, y_m), \quad where \quad x_l = lH, y_m = mH ,. \tag{43}$$

	N=100 S=1	N=200 s=1	 s=2	 s=3	N=500 s=1
K=10	$0.14\,10^{-09}$	$0.809\,10^{+67}$	8.079	$0.978\,10^{-13}$	
K=100	0.9430^{-03}	$0.80\,10^{+33}$	$0.223\,10^{-03}$		$0.155\,10^{-12}$
K=1000	0.0414	0.0141	$0.945\,10^{-03}$		$0.464\,10^{-05}$

Table 4. Values of $\max\limits_{i,j}\left|(u_{ij} - u^н_{ij})/u^н_{ij}\right|$ at $t = 0.199$, obtained by solving Problem 3 with using formula (43).

	N=100 s=1	 s=2	N=200 s=1	N=500 s=1	 s=2	 s=3
K=10	$0.166\;10^{+51}$	$0.504\,10^{-03}$	$0.179\,10^{-12}$			
K=100	61.98	$0.260\,10^{-03}$	$0.206\,10^{-06}$	$0.409\,10^{+147}$	3.61	$0.267\,10^{-12}$
K=1000	0.123	0.0391	$0.862\,10^{-02}$	$0.401\,10^{+24}$	16.08	0.170

Table 5. Values of $\max\limits_{i,j}\left|(u_{ij} - u^н_{ij})/u^н_{ij}\right|$ at $t = 0.199$, obtained by solving Problem 4 with using formula (43).

Hereinafter s is the number of smoothing iterations. As may be seen from tables 4, 5 in case if the breaking point of heat conduction factor gets into the node with odd number on the fine grid, at use of one smoothing iteration there can be a loss of stability of the method, for preservation of stability of the method and reception of good accuracy we should carry out some smoothing iterations.

We consider the heat conduction factor in the nodes of the coarse grid by the formula:

$$k'_{lm} = \frac{1}{4}\{k_{ij} + \frac{1}{2}\left(k_{i+1,j} + k_{i,j-1} + k_{i-1,j} + k_{i,j+1}\right) + \\ \frac{1}{4}\left(k_{i+1,j+1} + k_{i-1,j+1} + k_{i+1,j-1} + k_{i-1,j-1}\right)\}, \quad i = 2l, j = 2m \tag{44}$$

or by Approximation 2.

Table 6 displays the relative error $\max\limits_{i,j,t}\left|(u_{ij} - u^н_{ij})/u^н_{ij}\right|$ at $0 < t \le 0.199, n \ge 2$ and $\max\limits_{i,j}\left|(u_{ij} - u^н_{ij})/u^н_{ij}\right|$ at $t = 0.199$, obtained at the solution of problem 4.

	N = 100 s = 1 s = 2	 s = 1	N = 200 s= 1 s = 2	 s = 1
K = 10	0.061 0.038	$0.143\,10^{-9}$	0.092 0.06	$0.164\,10^{-12}$
K = 100	0.048 0.019	$0.189\,10^{-2}$	0.077 0.033	$0.128\,10^{-3}$
K = 1000		0.0492	0.035 0.025	$0.397\,10^{-2}$

	N =500 s = 1	s = 2	s = 1
K = 10			
K = 100	0.045	0.026	$0.247\,10^{-12}$
K = 1000	0.037	0.013	$0.184\,10^{-04}$

Table 6. Values of $\max\limits_{i,j,t}\left|(u_{ij}-u_{ij}^{н})/u_{ij}^{н}\right|$ and $\max\limits_{i,j}\left|(u_{ij}-u_{ij}^{н})/u_{ij}^{н}\right|$ at $t=0.199$, obtained by solving Problem 4 with using formula (44).

As may be seen from the table 6 more accurate account of heat conduction factor values leads to the situation where for preservation of stability at the solution of problem 4, one smoothing iteration is sufficient, and for obtaining good accuracy one or two smoothing iterations are required. Similar results are obtained for problem 3.

In model Problem 5 heat conduction factor is defined by formula:

$$k=\begin{cases}100, if\ (x-0.5)^2+(y-0.5)^2<1/16,\\ 1, otherweise\end{cases}.$$

In model Problem 6 heat conduction factor is defined by formula::

$$k=\begin{cases}100\left(1+0.3\sin 10\pi y\cdot\sin 10\pi x\right), if\ (x-0.5)^2+(y-0.5)^2<1/16,\\ 1, otherwise,\end{cases}$$

In model Problem 7 heat conduction factor is defined by formula:

$$k=\begin{cases}100\left(1+0.3\sin 10\pi x\right), if\ (x-0.5)^2+(y-0.5)^2<1/16,\\ 1, otherwise.\end{cases}$$

The initial and boundary conditions are as in Problem 3. Approximation 2 was used to calculate heat conduction factor in the coarse-grid computations. The results are shown in Tables 7 and 8.

The approximation of heat conduction factor was carried out in the nodes of the coarse grid by Approximation 2.

Tables 7-9 displays the relative error $\max\limits_{i,j,t}\left|(u_{ij}-u_{ij}^{н})/u_{ij}^{н}\right|$ at $0<t\le 0.199, n\ge 2$ (first and second numbers) and $\max\limits_{i,j}\left|(u_{ij}-u_{ij}^{н})/u_{ij}^{н}\right|$ at $t=0.199$ (the third number), obtained at the solution of problems 5-7.

	N=100 S=1	s=2	s=1	N=200 s=1	s=2	s=1
K=10	0.045	0.031	$0.96\,10^{-9}$	0.03	0.021	$0.245\,10^{-10}$
K=50	0.026	0.014	$0.62\,10^{-5}$	0.021	0.011	$0.169\,10^{-8}$
K=100	0.02	0.007	$0.265\,10^{-3}$	0.016	0.007	$0.129\,10^{-6}$

	N=500 S=1 s=2	s=1
K=10		
K=50		
K=100	0.028 0.015	$0.333\,10^{-10}$

Table 7. Values of $\max\limits_{i,j,t}\left|(u_{ij}-u_{ij}^{н})/u_{ij}^{н}\right|$ and $\max\limits_{i,j}\left|(u_{ij}-u_{ij}^{н})/u_{ij}^{н}\right|$ at $t=0.199$, obtained by solving Problem 5.

	N=100 s=1 s=2	s=1	N=200 s=1 s=2	s=1
K=10	0.047 0.032	$0.108\,10^{-8}$	0.03 0.022	$0.259\,10^{-10}$
K=50	0.027 0.014	$0.137\,10^{-4}$	0.022 0.011	$0.203\,10^{-8}$
K=100	0.021 0.009	$0.313\,10^{-3}$	0.017 0.007	$0.170\ 10^{-6}$

	N=500 S=1 s=2	s=1
K=10		
K=50		
K=100	0.029 0.016	$0.353\,10^{-10}$

Table 8. Values of $\max\limits_{i,j,t}\left|(u_{ij}-u_{ij}^{н})/u_{ij}^{н}\right|$ and $\max\limits_{i,j}\left|(u_{ij}-u_{ij}^{н})/u_{ij}^{н}\right|$ at $t=0.199$, obtained by solving Problem6.

	N=100 S=1 s=2	s=1	N=200 s=1 s=2	s=1
K=10	0.044 0.030	$0.495\,10^{-6}$	0.03 0.021	$0.122\,10^{-4}$
K=50	0.031 0.015	$0.137\,10^{-4}$	0.024 0.016	$0.304\ 10^{-4}$
K=100	0.03 0.013	$0.41\,10^{-2}$	0.016 0.007	$0.126\ 10^{-4}$

	N=500 s=1 s=2	s=1
K=10		
K=50		
K=100	0.033 0.016	$0.133\,10^{-4}$

Table 9. Values of $\max\limits_{i,j,t}\left|(u_{ij}-u_{ij}^{н})/u_{ij}^{н}\right|$ and $\max\limits_{i,j}\left|(u_{ij}-u_{ij}^{н})/u_{ij}^{н}\right|$ at $t=0.199$, obtained by solving Problem 7

The results obtained demonstrate that good accuracy can be achieved by applying the proposed method to initial–boundary value problems for the heat equation with discontinuous coefficients.

For studying the influence of heat conduction factor jump value on the accuracy of the proposed method, calculations of problem 8 with small jump k and problem 9 with continuous heat conduction factor were carried out.

In Problem 8:

$$k=\begin{cases}1+0.3\sin 10\pi x, & if\,(x-0.5)^2+(y-0.5)^2<1/16,\\ 1, otherwise.\end{cases}$$

In Problem 9

$$k = 1 + 0.3\sin 10\pi x \ .$$

Tables 10 and 11 show the calculations' results: $\max\limits_{i,j,t}\left|(u_{ij} - u_{ij}^{н}) / u_{ij}^{н}\right|$ at $0 < t \le 0.199, n \ge 2$, (the first) and $\max\limits_{i,j}\left|(u_{ij} - u_{ij}^{н}) / u_{ij}^{н}\right|$ at $t = 0.199$ (the second)

The approximation of heat conduction factor was carried out in the nodes of the coarse grid by formula(44).

	N=100		N=200		N=500	
K=10	0.000313	$0.213\ 10^{-7}$	0.000197	$0.207\ 10^{-7}$		
K=100	0.000457	$0.979\ 10^{-6}$	0.000276	$0.951\ 10^{-7}$	0.0000898	$0.102\ 10^{-7}$

Таблица10 Values of $\max\limits_{i,j,t}\left|(u_{ij} - u_{ij}^{н}) / u_{ij}^{н}\right|$ and $\max\limits_{i,j}\left|(u_{ij} - u_{ij}^{н}) / u_{ij}^{н}\right|$ at $t = 0.199$, obtained by solving Problem 8 (s=1).

	N=100		N=200		N=500	
K=10	0.000243	$0.416\ 10^{-7}$	0.0000614	$0.207\ 10^{-7}$		
K=100	0.00029	$0.898\ 10^{-6}$	0.0000846	$0.3781\ 10^{-7}$	0.000024	$0.188\ 10^{-8}$

Table 11 Values of $\max\limits_{i,j,t}\left|(u_{ij} - u_{ij}^{н}) / u_{ij}^{н}\right|$ and $\max\limits_{i,j}\left|(u_{ij} - u_{ij}^{н}) / u_{ij}^{н}\right|$ at $t = 0.199$, obtained by solving Problem 9 (*s=1).*

To examine the dependence of accuracy on the value of the jump in thermal conductivity, we computed Problem 8, with a relatively small jump *k*, and Problem 9, with continuous thermal conductivity. Comparing the corresponding errors shown in Tables 10 and 11, respectively, we see that higher accuracy is achieved in Problems 8 and 9 as compared to Problem 7.

6. CONCLUSION

The new effective solution algorithm for diffusion-type equations has been created. On an example of some model problems it has been demonstrated both theoretically and numerically that the proposed method has stability and accuracy inherent in the implicit scheme. The created algorithm allows reducing considerably the volume of arithmetic work on each time level in comparison with the use of the implicit scheme.

10. REFERENCES

[1] A.A. Samarsky "Difference scheme theory" *Nauka*, (1989).

[2] N.S. Bahvalov, N.P. Zhidkov and G.M. Kobelkov, "Numerical methods", *Nauka*, (1987).

[3] Fedorenko R.P., “A relaxation method for solving difference elliptic equations”,*Zh. Vychisl. Mat. Mat. Fiz*., Vol.1, N 5, 922-927 (1961).

[4] S.K. Godunov, V.S.Ryabenkiy, ”Difference schemes”, *Nauka,* (1977).

[5] R. Rihtmaer, K.Morton, “Difference methods for solving of boundary value problem”, Mir, (1972).

[6] M.E. Ladonkina, O.Yu. Milyukova, V.F. Tishkin, “A numerical algorithm for diffusion-type equations based on the multigrid methods”, *Mat. Model*., Vol.19, 71-89, (2007).

[7] A.A. Samarsky, Ye.S. Nikolayev, “Methods for solving finite-difference equation”, Nauka, (1978).

[8] I. Gustafsson, *A Class of First Order Factorization Methods.* BIT. V.18. P.142-156, (1978).