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ON THE FORM OF THE HYDRODYNAMICS EQUATIONS

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Abstract. In theoretical studying of applied problems the conservative form of the hydrodynamics equations is the most commonly used. In this case the equations have the divergent form and express directly the corresponding laws of conservation (of mass, momentum and energy) In some cases one should pay attention to non-divergent (characteristic) form of the hydrodynamics equations which is connected with the representation derived via differentiating the convective transport terms. This paper presents a new form of the hydrodynamics equations which is characterized by writing convection terms in the skew-symmetric form. New quantities — so called SD–variables (**S**quare root from **D**ensity) — based on using not the density but the square root from density are used as unknowns. Physical and mathematical arguments in favor of using this form of the hydrodynamics equations are discussed.

1. INTRODUCTION

Differential equations of hydrodynamics can be written in various forms. They express laws of conservation of mass, momentum and energy.[1,2] Such an interpretation is the most clear when the equations of hydrodynamics are written in the conservative form. In this case terms of the equations which are responsible for convective transport have the divergent form. Straighforward integration of the equations over a fluid volume leads us to the integral form of the corresponding conservation law.

The non-conservative form of the hydrodynamics equations is derived via simple transformations. In this case the convective transport is written in the non-divergent (characteristic) form. The non-conservative form is related to some properties of the solution, such as fulfillment of the maximum principle and receiving uniform estimates for the solution. A more detailed discussion of these problems can be found in books[3,4] where model equations of convection-diffusion are considered.

In increasing frequency the so-called symmetric form of the motion equations is used for the numerical solution of problems of incompressible fluid dynamics (see e.g.[5,6]). In this case the convective terms are written as the half-sum of the convective terms in conservative and non-conservative forms: that's why this form is called *symmetric*. The main advantage of this way of writing is, that it is the simplest way to satisfy the condition of energy neutrality of the momentum equation

conventional signs

$\mathcal{C}$	the operator of convective transport
D	diffusion coefficient
e	internal energy
E	identity tensor
L^1, L^2	function spaces
$\mathbf{n}$	outer normal to $\partial\Omega$
N	viscous stress tensor
p	pressure
s	square root from the density
S	strain rate tensor
t	time
T	the final time moment
$\mathbf{u}$	the velocity vector
u_1, u_2, u_3	velocity components
$\mathbf{v} = (v_1, v_2, v_3)$	desired vector function
$\mathbf{w}$	new unknown vector function
$\mathbf{x} = (x_1, x_2, x_3)$	point of space
x_1, x_2, x_3	Cartesian coordinates

Greek letters

ζ	new unknown scalar function
μ	dynamic viscosity
φ	desired scalar function
ϱ	fluid density
Ω	flow domain
$\partial\Omega$	domain boundary

indexes

$i = 1, 2, 3$	a number of independent variable, vector components

— the convective terms don't contribute to the kinetic energy. The mathematical expression of this fact is the skew–symmetric property of the operator of convective transport in common Hilbert space L^2.[4] Note, that the skew–symmetric property takes place not only for incompressible flows. Subject to that, this form of writing of the convective terms is called also[7,8] skew-symmetric.

To ensure on the discrete level the most simple fulfilment of the skew-symmetric property, we should add in the incompressible Navier–Stokes equation (written in the non-conservative form) a regularizing term proportional to the velocity divergence.[9] This way to present the convective term can be interpreted as another differential writing of the operator of convective transport in symmetric (skew–symmetric) form. We'd also point on two-dimensional hydrodynamic problem in variables *stream function, vorticity*: in these problems a form of the convective terms always attracts great attention.[10] In papers[11,12] a special form of convective terms of the stream function transport that ensures the fulfilment of skew–symmetric property was suggested.

Various forms of the hydrodynamics equations are completely equivalent on the differential level. That's why a transition from one form of equations to another isn't of any significance in its own right. Then we proceed to the numerical solution of applied problems, our choice of way of writing the original equations is the problem of great importance. There is no equivalence on the discrete level (after approximation in time and in space). Therefore properties of the differential problem don't remain valid for the discrete problem. So we have to choose the initial differential equations purposefully in order to preserve main and sacrifice minor properties of the differential problem.

In this paper a new form of the hydrodynamics equations is suggested which is characterized by writing the convective terms in the skew–symmetric form. In this case the fulfilment of corresponding conservation laws is connected to a priori estimates in the most suitable for research Hilbert space L^2. Particularly, in linear case, corresponding analogs of these a priori estimates on a discrete level ensure the stability of an approximate solution and its convergence to the exact solution of the differential problem. In a new form of hydrodynamics equations we suggest new variables as unknowns which are based not on the density, but on the square root from the density. We call them SD–variables: (**S**quare root from **D**ensity).

A few words on the content. In section 2 there is presented a basic system of the hydrodynamics equations consisting of two equations. The continuity equation and the equation of convection–diffusion for a scalar quantity are considered as a prototype of the system of hydrodynamic equations taking into account compressibility effects. The last equation can be regarded particularly as an individual component of a vector value. The equation of convection–diffusion taking account of the continuity equation can be written in the conservative or non-conservative form. In section 3 a priori estimates are given for a model problem in functional spaces L^1 L^2, which express the corresponding laws of conservation. The symmetric form of the basic hydrodynamics equations is described in section 4. New unknown variables based on using the square root from the density are suggested. In SD–variables the basic system of hydrodynamic equations has the most suitable form for constructing discrete models and studying them.

2. BASIC SYSTEM OF THE HYDRODYNAMICS EQUATIONS

The system of hydrodynamics equations includes, first of all, the scalar equation of continuity and the vector equation of motion. In more common cases there can be several motion equations and continuity equations — models of multicomponent media. Furthermore, the system of equations can be added with an energy equation. Usually, the following scalar equation of convection-diffusion performs as the basic equation in continuum mechanics and heat and mass transfer (for example, see[13, 14])

$$\frac{\partial(\varrho\varphi)}{\partial t} + \operatorname{div}(\varrho \mathbf{u}\varphi) = \operatorname{div}(D \operatorname{grad} \varphi). \tag{1}$$

This equation is written in the conservative (divergent) form. With regard to equation (1) a number of problems are discussed, such as construction of space and time approximations, investigation of convergence of the approximate solution to the exact solution of the corresponding boundary-value problem.[4,15]

The main peculiarities of the system of fluid dynamic equations are demonstrated in the system of two scalar equations, which includes not only (1) but the continuity equation too

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \mathbf{u}) = 0. \tag{2}$$

Just this system of equations (1), (2) we'll call the basic system of scalar hydrodynamic equations.

When describing transport phenomena in continuum mechanics, features of transport of scalar values are represented in equation (2). Concerning vector fields, the coordinate representation can be unsuitable. So we can supplement our system of equations (1), (2) with the vector equation of convection-diffusion

$$\frac{\partial (\varrho \mathbf{v})}{\partial t} + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{v}) = \operatorname{div}(D \operatorname{grad} \mathbf{v}). \tag{3}$$

We'll call this system of equations (1) - (3) the basic system of hydrodynamics equations.

Taking into account

$$\operatorname{div}(\varrho \mathbf{u} \varphi) = \varphi \operatorname{div}(\varrho \mathbf{u}) + \varrho \mathbf{u} \cdot \operatorname{grad} \varphi$$

and continuity equation (2), we come to

$$\varrho \frac{\partial \varphi}{\partial t} + \varrho \mathbf{u} \cdot \operatorname{grad} \varphi = \operatorname{div}(D \operatorname{grad} \varphi). \tag{4}$$

Similarly we can rewrite equation (3):

$$\varrho \frac{\partial \mathbf{v}}{\partial t} + \varrho \mathbf{u} \cdot \operatorname{grad} \mathbf{v} = \operatorname{div}(D \operatorname{grad} \mathbf{v}). \tag{5}$$

Equations (4), (5) are written in non-conservative (non-divergent) form. Note, that continuity equation (2) can't be written in non-conservative form. Thereby, the basic system of hydrodynamics equations can be written in conservative (1) - (3) or partially non-conservative form (2), (4), (5). Only for an incompressible medium, where equation (2) takes on form

$$\operatorname{div} \mathbf{u} = 0,$$

it's possible to speak about non-conservative form of equations.

3. THE SOLUTION PROPERTIES

Let state a model boundary-value problem for equations (1) - (3) in a bounded region Ω with a boundary $\partial\Omega$. Desired values – the density $\varrho(\mathbf{x}, t)$, the scalar function $\varphi(\mathbf{x}, t)$ and the vector function $\mathbf{v}(\mathbf{x}, t)$. Then we supplement it with necessary boundary and initial conditions. Let the normal component of velocity vector at the boundary is equal to zero:

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad \mathbf{x} \in \partial\Omega. \tag{6}$$

For hydrodynamics problems condition (6) is the standard one and drscribes no-permeability of a flow.

With the assumption of (6), boundary conditions are not stated for continuity equation(2). For equations (1) and (3), required boundary conditions are determined by the operator of diffusion. For simplicity let confine ourselves to examine homogeneous boundary Dirichlet conditions for desired scalar and vector functions:

$$\varphi(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \partial\Omega, \tag{7}$$

$$\mathbf{v}(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \partial\Omega. \tag{8}$$

Suppose we are looking for the solution of time-dependent equations (1) - (3) with $0 < t \le T$. An initial state is determined by the following conditions

$$\varrho(\mathbf{x}, 0) = \varrho_0(\mathbf{x}), \tag{9}$$

$$\varphi(\mathbf{x}, 0) = \varphi_0(\mathbf{x}), \tag{10}$$

$$\mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}). \tag{11}$$

After integration (2) over Ω, with regard to condition (6) we come to

$$\frac{\partial}{\partial t} \int\limits_{\Omega} \varrho(\mathbf{x}, t)\, d\mathbf{x} = 0.$$

This condition expresses the mass conservation property. Taking into consideration the starting condition (9), we'll get

$$\int\limits_{\Omega} \varrho(\mathbf{x}, t)\, d\mathbf{x} = \int\limits_{\Omega} \varrho_0(\mathbf{x})\, d\mathbf{x}, \quad 0 < t \le T. \tag{12}$$

For the continuity equation the maximum principle is satisfied.[16,17] That means, that under the condition $\varrho_0(\mathbf{x}) \ge 0$ at any other time moment the density will be non-negative. Thereby we can regard equality (12) as a priori estimate in $L^1(\Omega)$:

$$\|\varrho(\mathbf{x}, t)\|_1 = \|\varrho_0(\mathbf{x})\|_1, \quad 0 < t \le T, \tag{13}$$

where norm in $L^1(\Omega)$ of function $\theta(\mathbf{x})$ is defined according to rule

$$\|\theta(\mathbf{x})\|_1 = \int\limits_{\Omega} |\theta(\mathbf{x})|\, d\mathbf{x}.$$

After integration of (1) we come to

$$\frac{\partial}{\partial t} \int\limits_{\Omega} \varrho(\mathbf{x}, t)\varphi(\mathbf{x}, t)\, d\mathbf{x} = \int\limits_{\partial\Omega} D \frac{\partial \varphi}{\partial n}\, d\mathbf{x}. \tag{14}$$

In a general case, with equality (14), we can't associate directly any a priori estimate in $L^1(\Omega)$.

For equation (1) a priori estimate in $L^2(\Omega)$ seems to be more natural. To derive it, we'll summarize equations (1) and (4), multiply by φ and integrate over Ω. Taking into account equalities

$$\varphi \frac{\partial (\varrho\varphi)}{\partial t} + \varrho\varphi \frac{\partial \varphi}{\partial t} = \frac{\partial (\varrho\varphi^2)}{\partial t},$$

$$\varphi \operatorname{div}(\varrho \mathbf{u} \varphi) + \varrho \varphi \mathbf{u} \cdot \operatorname{grad} \varphi = \operatorname{div}(\varrho \mathbf{u} \varphi^2)$$

and boundary condition (10), we'll get the following equality

$$\frac{\partial}{\partial t} \int_\Omega \varrho(\mathbf{x}, t) \varphi^2(\mathbf{x}, t)\, d\mathbf{x} + 2 \int_\Omega D (\operatorname{grad} \varphi)^2\, d\mathbf{x} = 0. \tag{15}$$

Like (14), equality (15) is naturally interpreted as the corresponding conservative law. In case (14) we mean conservation of $\varrho\varphi$, and in case (15) — $\varrho\varphi^2$.

Let define the norm in $L^2(\Omega)$ of function $\theta(\mathbf{x})$ according to

$$\|\theta(\mathbf{x})\|_2 = \left(\int_\Omega \theta^2(\mathbf{x})\, d\mathbf{x} \right)^{1/2}.$$

Then we can write equation (15) in the following form

$$\frac{\partial}{\partial t} \|\varrho^{1/2} \varphi\|^2 + 2 \int_\Omega D (\operatorname{grad} \varphi)^2\, d\mathbf{x} = 0. \tag{16}$$

From (16) directly follows a prior estimate

$$\|\varrho^{1/2} \varphi\|_2 \le \|\varrho_0^{1/2} \varphi_0\|_2, \quad 0 < t \le T. \tag{17}$$

Subject to (17) let rewrite equality (12) as follows

$$\|\varrho^{1/2}\|_2 = \|\varrho_0^{1/2}\|_2, \quad 0 < t \le T. \tag{18}$$

Similarly for equations (3), (5) we obtain the estimate

$$\|\varrho^{1/2} \mathbf{v}\|_2 \le \|\varrho_0^{1/2} \mathbf{v}_0\|_2, \quad 0 < t \le T. \tag{19}$$

Thereby for the system of equations (1) - (2), supplemented with conditions (6) - (10), take place a prior estimates (17) - (19) in Hilbert space $L^2(\Omega)$. Just this estimates should be regarded as fundamental for the basic system of equations in hydrodynamics. When constructing discrete analogs one should focus on fulfilment of such estimates also for the approximate solution, that can be satisfied via choosing the appropriate form of basic fluid dynamic equations.

4. THE SYMMETRIC FORM

When constructing discrete analogs, it is necessary to focus on fulfilment of such estimates also for the approximate solution, that are analogous to (17) - (19) and that also provide a basis for estimation of convergence rate of th approximate solution to the exact one. To solve this problem, one needs to choose the appropriate form of the basic fluid dynamic equations.

Let write the operator of convective transport[4,15] in the symmetric form:

$$\mathcal{C}\theta = \frac{1}{2} \operatorname{div}(\mathbf{u}\theta) + \frac{1}{2} \mathbf{u} \cdot \operatorname{grad} \theta, \tag{20}$$

i.e., as the half-sum of the operators of convective transport in divergent (conservative) and non-divergent (non-conservative) forms. With restrictions (6) we get

$$\int\limits_{\Omega} \theta \mathcal{C} \theta \, d\mathbf{x} = 0.$$

That means, that the operator $\mathcal{C}$ is skew-symmetric in $L^2(\Omega)$ ($\mathcal{C} = -\mathcal{C}^*$). With regard to this property this form of writing of convective transport (20) sometimes is called skew-symmetric.

In the basic system of fluid dynamics equations (1) - (3) instead of $\varrho, \varphi, \mathbf{v}$ we introduce new desired functions:

$$s = (\varrho)^{1/2}, \quad \zeta = (\varrho)^{1/2} \varphi, \quad \mathbf{w} = (\varrho)^{1/2} \mathbf{v}. \tag{21}$$

The speciality of this unknowns consists in using not the density ϱ itself, but the square root from density $s = (\varrho)^{1/2}$. That is why we speak about SD - variables (**S**quare root from **D**ensity).

For new unknowns the system of equations (1) - (3) is rewritten in the following way

$$\frac{\partial s}{\partial t} + \frac{1}{2} \operatorname{div}(\mathbf{u} s) + \frac{1}{2} \mathbf{u} \cdot \operatorname{grad} s = 0, \tag{22}$$

$$\frac{\partial \zeta}{\partial t} + \frac{1}{2} \operatorname{div}(\mathbf{u} \zeta) + \frac{1}{2} \mathbf{u} \cdot \operatorname{grad} \zeta = \frac{1}{s} \operatorname{div} \left(D \operatorname{grad} \left(\frac{\zeta}{s} \right) \right), \tag{23}$$

$$\frac{\partial \mathbf{w}}{\partial t} + \frac{1}{2} \operatorname{div}(\mathbf{u} \otimes \mathbf{w}) + \frac{1}{2} \mathbf{u} \cdot \operatorname{grad} \mathbf{w} = \frac{1}{s} \operatorname{div} \left(D \operatorname{grad} \left(\frac{\mathbf{w}}{s} \right) \right). \tag{24}$$

In this case in all three equations the convective terms are written in the same symmetric form.

The corresponding a priori estimates (17) - (19) take on the simplest form:

$$\|s\|_2 = \|\varrho_0^{1/2}\|_2, \tag{25}$$

$$\|\zeta\|_2 \le \|\varrho_0^{1/2} \varphi_0\|_2, \tag{26}$$

$$\|\mathbf{w}\|_2 \le \|\varrho_0^{1/2} \mathbf{v}_0\|_2, \quad 0 < t \le T. \tag{27}$$

What is important, estimates (25) - (27) are obtained for each particular equation without involving other equations, on the basis of the skew-symmetric property of convective transport operator (20). As for discrete analogs of equations (17) - (19), a priori estimates like (25) - (27) for the approximate solution can be obtained very easy and in most appropriate norms.

5. NAVIER-STOKES EQUATIONS

As a typical example of using of new variables we consider the Navier-Stokes equations for a viscous compressible medium, which express conservations laws for mass, momentum and energy. The continuity equation has form (2). The equation of motion usually is written in conservative form

$$\frac{\partial (\varrho \mathbf{u})}{\partial t} + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) = \operatorname{div} \mathsf{N} - \operatorname{grad} p. \tag{28}$$

Here

$$\mathsf{N} = -\frac{2}{3}\mu \operatorname{div} \mathbf{u}\, \mathsf{E} + 2\mu \mathsf{S},$$

And for S the coordinate representation takes place

$$\mathsf{S}_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Let present also the equation of energy

$$\frac{\partial(\varrho e)}{\partial t} + \operatorname{div}(\varrho \mathbf{u} e) = \operatorname{div}(k \operatorname{grad} T) + p \operatorname{div} \mathbf{u} + \mathsf{N} : \operatorname{grad} \mathbf{u}. \quad (29)$$

The term $\mathsf{N} : \operatorname{grad} \mathbf{u}$ describes heat dissipation due to the fluid viscosity, and $\mathsf{N} : \operatorname{grad} \mathbf{u}$ — scalar product of tensors:

$$\begin{aligned} \mathsf{N} : \operatorname{grad} \mathbf{u} = \quad & N_{xx}\frac{\partial u_x}{\partial x} + N_{xy}\frac{\partial u_x}{\partial y} + N_{xz}\frac{\partial u_x}{\partial z} + \\ & N_{yx}\frac{\partial u_y}{\partial x} + N_{yy}\frac{\partial u_y}{\partial y} + N_{yz}\frac{\partial u_y}{\partial z} + \\ & N_{zx}\frac{\partial u_z}{\partial x} + N_{zy}\frac{\partial z_y}{\partial y} + N_{zz}\frac{\partial u_z}{\partial z}. \end{aligned}$$

Let us introduce the following unknowns as desired variables:

$$s = (\varrho)^{1/2}, \quad \mathbf{w} = (\varrho)^{1/2}\mathbf{u}, \quad \zeta = (\varrho)^{1/2} e. \quad (30)$$

In variables (30) the system of the Navier-Stokes equations (2), (28), (29) has the following form:

$$\frac{\partial s}{\partial t} + \frac{1}{2}\left(\operatorname{div}\left(\frac{\mathbf{w}}{s} s\right) + \frac{\mathbf{w}}{s} \cdot \operatorname{grad} s\right) = 0, \quad (31)$$

$$\frac{\partial \mathbf{w}}{\partial t} + \frac{1}{2}\left(\operatorname{div}\left(\frac{\mathbf{w}}{s} \otimes \mathbf{w}\right) + \frac{\mathbf{w}}{s} \cdot \operatorname{grad} \mathbf{w}\right) = \frac{1}{s} \operatorname{div} \mathsf{N} - \frac{1}{s} \operatorname{grad} p, \quad (32)$$

$$\begin{aligned} & \frac{\partial \zeta}{\partial t} + \frac{1}{2}\left(\operatorname{div}\left(\frac{\mathbf{w}}{s}\zeta\right) + \frac{\mathbf{w}}{s} \cdot \operatorname{grad} \zeta\right) = \\ & \frac{1}{s} \operatorname{div}(k \operatorname{grad} T) + \frac{p}{s} \operatorname{div}\left(\frac{\mathbf{w}}{s}\right) + \frac{1}{s}\mathsf{N} : \operatorname{grad}\left(\frac{\mathbf{w}}{s}\right). \end{aligned} \quad (33)$$

The system of equations (31) - (33) needs to be supplemented by some equations of state. What's significant, in variables (30) the convective terms are written in the symmetric form.

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